A Trust Funnel Algorithm for Nonconvex Equality Constrained Optimization with $O(\epsilon^{-3/2})$ Complexity

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Outline

Motivation

Proposed Algorithm

Theoretical Results

Numerical Results

Summary
## Outline

- **Motivation**
- **Proposed Algorithm**
- **Theoretical Results**
- **Numerical Results**
- **Summary**
Consider nonconvex equality constrained optimization problems of the form

\[
\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t. } c(x) = 0.
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) and \( c : \mathbb{R}^n \to \mathbb{R}^m \) are twice continuously differentiable.

- We are interested in algorithm worst-case iteration / evaluation complexity.
- Constraints are not necessarily linear!
Algorithms for equality constrained (nonconvex) optimization

Sequential Quadratic Programming (SQP) / Newton’s method

Trust Funnel; Gould & Toint (2010)

Short-Step ARC; Cartis, Gould, & Toint (2013)
Sequential Quadratic Programming (SQP) / Newton’s method

- **Global convergence**: globally convergent (trust region/line search)

Trust Funnel; Gould & Toint (2010)

- **Global convergence**: globally convergent

Short-Step ARC; Cartis, Gould, & Toint (2013)

- **Global convergence**: globally convergent
Sequential Quadratic Programming (SQP) / Newton’s method

- **Global convergence**: globally convergent (trust region/line search)
- **Worst-case complexity**: No proved bound

Trust Funnel; Gould & Toint (2010)

- **Global convergence**: globally convergent
- **Worst-case complexity**: No proved bound

Short-Step ARC; Cartis, Gould, & Toint (2013)

- **Global convergence**: globally convergent
- **Worst-case complexity**: $O(\epsilon^{-3/2})$
Short-Step ARC

\[ (0, f^*) \to \|c(x)\|^2 \]

\[ f(x) \]

\[ (0, f^*) \]
Short-Step ARC

\[ f(x) \]

\[ (0, f^*) \]

\[ \|c(x)\|^2 \]
Short-Step ARC

\[ f(x) \]

\[ \| c(x) \|^2 \]

\[(0, f^*)\]
Main Concerns

- Completely ignores the objective function during the first phase
- **Question**: Can we do better?
Main Concerns

- Completely ignores the objective function during the first phase
- **Question:** Can we do better?
- **Yes! (?)**
- First, rather than two-phase approach that ignores objective in phase 1, wrap in a **trust funnel** framework that observes objective in both phases.
- Second, consider **TRACE** method for unconstrained nonconvex optimization
Outline

Motivation

Proposed Algorithm

Theoretical Results

Numerical Results

Summary
SQP “core”

- Given $x_k$, find $s_k$ as a solution of

$$\min_{s \in \mathbb{R}^n} f_k + g_k^T s + \frac{1}{2} s^T H_k s$$

s.t. $c_k + J_k s = 0$

Issues:

- $H_k$ might not be positive definite over $\text{Null}(J_k)$.
- Trust region!... but constraints might be incompatible.
Step decomposition

\[ c_k + J_k s = 0 \]
Step decomposition
Step decomposition

\[ c_k + J_k s = c_k + J_k s_k^n \]
Trust funnel basics

Step decomposition approach:

- First, compute a *normal step* toward minimizing constraint violation

\[
v(x) = \frac{1}{2} \|c(x)\|^2 \Rightarrow \min_{s^n \in \mathbb{R}^n} m_k^v(s^n) \quad \text{s.t. } \|s^n\| \leq \delta_k^v
\]

- Second, compute multipliers \(y_k\) (or take from previous iteration).

- Third, compute a *tangential step* toward optimality:

\[
\min_{s^t \in \mathbb{R}^n} m_k^f(s_k^n + s^t) \quad \text{s.t. } J_k s^t = 0, \|s_k^n + s^t\| \leq \delta_k^f.
\]
Main idea

Two-phase method combining trust funnel and TRACE.

- Trust funnel for globalization
- TRACE for good complexity bounds

Phase 1 towards feasibility, two types of iterations:

- F-ITERATIONS improve objective and reduce constraint violation.
- V-ITERATIONS reduce constraint violation.

Our algorithm vs. basic trust funnel

- modified F-ITERATION conditions and a different funnel updating procedure
- uses TRACE ideas (for radius updates) instead of tradition trust region
- after getting approximately feasible, switches to “phase 2”.
Our algorithm-Illustration

\[ f(x) \]

\[ (0, f^*) \]

\[ \| c(x) \|^2 \]
Our algorithm-Illustration

\[ f(x) \]

\[ \|c(x)\|^2 \]

\((0, f^*)\)
Our algorithm-Illustration
Our algorithm-Illustration

\[ (0, f^*) \]

\[ \|c(x)\|^2 \]
Our algorithm-Illustration

\[ f(x) \quad \|c(x)\|^2 \]

\[(0, f^*)\]
Our algorithm-Illustration

\[ (0, f^*) \rightarrow \| c(x) \|^2 \]
Our algorithm-Illustration

\[ f(x) \]

\[ (0, f^*) \]

\[ \| c(x) \|^2 \]
Our algorithm-Illustration
Our algorithm-Illustration

\[ f(x) \]

\[ (0, f^*) \]

\[ \|c(x)\|^2 \]
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Motivation

Proposed Algorithm

Theoretical Results

Numerical Results

Summary
Phase 1

Recall that $\nabla v(x) = J(x)^T c(x)$ and define the iteration index set

$$\mathcal{I} := \{k \in \mathbb{N} : \|J_k^T c_k\| > \epsilon_v\}.$$

**Theorem**

For any $\epsilon_v \in (0, \infty)$, the cardinality of $\mathcal{I}$ is at most $K(\epsilon_v) \in \mathcal{O}(\epsilon_v^{-3/2})$:

- $\mathcal{O}(\epsilon_v^{-3/2})$ successful steps and
- finite contraction and expansion steps between successful steps.

**Corollary**

If $\{J_k\}$ have full row rank with singular values bounded below by $\xi \in (0, \infty)$, then

$$\mathcal{I}_c := \{k \in \mathbb{N} : \|c_k\| > \epsilon_v/\xi\}$$

has cardinality $\mathcal{O}(\epsilon_v^{-3/2})$. 
Options for phase 2:
- trust funnel method (no complexity guarantees) or
- “target-following” approach similar to Short-Step ARC to minimize

\[ \Phi(x, t) = \|c(x)\|^2 + |f(x) - t|^2. \]

**Theorem**

For \( \epsilon_f \in (0, \epsilon_v^{1/3}] \), the number of iterations until

\[ \|g_k + J_k^T y\| \leq \epsilon_f \|(y_k, 1)\| \text{ or } \|J_k^T c_k\| \leq \epsilon_f \|c_k\| \]

is \( \mathcal{O}(\epsilon_f^{-3/2} \epsilon_v^{-1/2}) \).

Same complexity as Short-Step ARC:
- If \( \epsilon_f = \epsilon_v^{2/3} \), then overall \( \mathcal{O}(\epsilon_v^{-3/2}) \)
- If \( \epsilon_f = \epsilon_v \), then overall \( \mathcal{O}(\epsilon_v^{-2}) \)
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Motivation

Proposed Algorithm

Theoretical Results

Numerical Results

Summary
Implementation

MATLAB implementation:

- Phase 1: our algorithm vs. one doing V-iteration only
- Phase 2: trust funnel method [Curtis, Gould, Robinson, & Toint (2016)]

Termination conditions:

- Phase 1:
  \[ \|c_k\|_\infty \leq 10^{-6} \max\{\|c_0\|_\infty, 1\} \quad \text{or} \quad \left\{ \begin{array}{l}
  \|J_k^T c_k\|_\infty \leq 10^{-6} \max\{\|J_0^T c_0\|_\infty, 1\} \\
  \text{and} \quad \|c_k\|_\infty > 10^{-3} \max\{\|c_0\|_\infty, 1\}
\end{array} \right. \]

- Phase 2
  \[ \|g_k + J_k^T y_k\|_\infty \leq 10^{-6} \max\{\|g_0 + J_0^T y_0\|_\infty, 1\}. \]
Test set

Equality constrained problems (190) from CUTEst test set:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>78</td>
<td>constant (or null) objective</td>
</tr>
<tr>
<td>60</td>
<td>time limit</td>
</tr>
<tr>
<td>13</td>
<td>feasible initial point</td>
</tr>
<tr>
<td>3</td>
<td>infeasible phase 1</td>
</tr>
<tr>
<td>2</td>
<td>function evaluation error</td>
</tr>
<tr>
<td>1</td>
<td>small stepsizes (less than $10^{-40}$)</td>
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</table>

Remaining set consists of 33 problems.
<table>
<thead>
<tr>
<th>Problem</th>
<th>n</th>
<th>m</th>
<th>#V</th>
<th>#F</th>
<th>( | g + J^T y | )</th>
<th>( | f + J^T y | )</th>
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</thead>
<tbody>
<tr>
<td>Phase 1</td>
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<td>0</td>
<td>199</td>
<td>0</td>
<td>1</td>
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<td>47.79e-01</td>
<td>0</td>
<td>0</td>
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<tr>
<td>TF-V-only</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Phase 1</td>
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</tbody>
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**Phase 1**

**TF**

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<table>
<thead>
<tr>
<th>Problem</th>
<th>$n$</th>
<th>$m$</th>
<th>#V</th>
<th>#F</th>
<th>$f$</th>
<th>$|g + J^T y|$</th>
<th>#V</th>
<th>#F</th>
<th>$f$</th>
<th>$|g + J^T y|$</th>
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<tbody>
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<td>+4.84e-03</td>
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<td>0</td>
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<td>0</td>
<td>16</td>
<td>+3.34e+01</td>
</tr>
</tbody>
</table>

A Trust Funnel Algorithm for Nonconvex Equality Constrained Optimization
Summary of results

Our algorithm, at the end of phase 1

- for 26 problems, reaches a smaller function value
- for 6 problems, reaches the same function value

Total number of iterations of our algorithm

- for 18 problems is smaller
- for 8 problems is equal
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Summary
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- Proposed an algorithm for equality constrained optimization
- Trust funnel algorithm with improved complexity properties
- Promising performance in practice based on our preliminary numerical experiment
- A step toward practical algorithms with good iteration complexity