Finding Infimum Point with Respect to the Second Order Cone

In honor of Don Goldfarb

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The Dual Simplex Method for a Special SOCP Problem

The Simplex Method has been extended to convex Quadratic Programming decades ago (Franke-Wolfe 55)

(Goldfarb-Idnani 83) gave a *practical dual* algorithm (our research is inspired partly by their work)

The simplex method can be extended to a large class of *LP-Type problems* (Matousek, Sharir, Welzl 96)

Competitiveness and contrast to Interior Point Methods

Simplex vs Interior point methods, why simplex?

Reminder: For linear optimization:

- Interior point (IP) methods usually have to solve a full-fledged linear system per iteration, but have a small number of iterations
- In the simplex method a low rank update of a previously solved system must be found, but the number of iterations is large
- IP methods are better for parallel implementation, and sparse systems
- Simplex is better for warm-start, and for cases where constraints arrive in a stream
- Dual simplex is also generally more suitable for branch and bound and similar procedures

A Similar situation exists for problems more general than linear optimization

Infimum with respect to the Second-Order cone

Let $\mathcal{Q}$ be the second-order cone $\mathcal{Q} := \left\{x = (x_0; \bar{x}) \in \mathbb{R}^d : \|\bar{x}\|_2 \leq x_0\right\}$

We define the *infimum of a set of points* $\mathcal{P} = \{p_1, \ldots, p_m\} \subset \mathbb{R}^d$ *with respect to* $\mathcal{Q}$ as:

$$\begin{aligned} \mathrm{Inf}_{\mathcal{Q}}(\mathcal{P}) := \max_x \; & x_0 \quad (= \langle e_0, x \rangle) \\ \text{s.t.} \; & x \preceq_{\mathcal{Q}} p_i, \; i = 1, ..., m \end{aligned}$$

with $e_0 = (1, 0, \ldots, 0)^\top$.

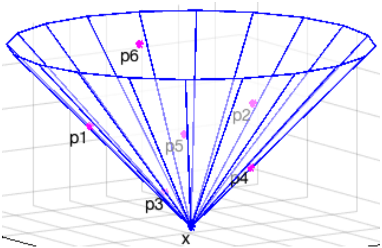


Fig 1. Example in $\mathbb{R}^3$ with 6 points.

It does not seem that this problem is a QP.

Equivalence to the Smallest Enclosing Ball of Balls

Lemma

$B(c_1, r_1) \subseteq B(c_2, r_2)$ *iff* $\|c_2 - c_1\| \leq r_2 - r_1$.

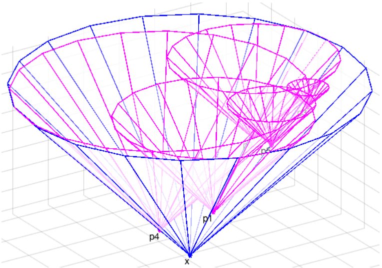


Fig 2. Consider a SOC with vertex at each p_i.

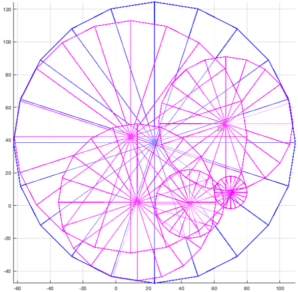


Fig 3. View from the top.

The smallest ball containing a set of balls:

$$\begin{aligned}&\max_{x} x_0 \, (= \text{radius})\\ &\text{s.t. } \|\bar{p}_i - \bar{x}\| \leq p_{i0} - x_0, \, i = 1, ..., m\end{aligned}$$

But $\|\bar{p}_i - \bar{x}_i\| \leq p_{i0} - x_0 \quad \Leftrightarrow \quad \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} \preceq_{\mathcal{Q}} \begin{pmatrix} p_{i0} \\ \bar{p}_i \end{pmatrix}$

The smallest enclosing ball of balls is an *"LP-type"* problem (Matoušek, Sharir & Welzl (1996))

Previous work: Megiddo (1989); Welzl (1991); Chazelle and Matoušek (1996); Bădoiu et al. (2002); Fischer and Gärtner (2003); Kumar et al. (2003); Zhou et al (2005).

Duality and Complementary Slackness

Dual problem:

$$\min_{y} \sum_{i=1}^{m} \langle p_i, y_i \rangle$$

$$\text{s.t.} \quad \sum_{i=1}^{m} y_i = e_0$$

$$y_i \succeq_{\mathcal{Q}} 0, \quad i = 1, \ldots, m$$

Complementary slackness:

$$\langle p_i - x, y_i \rangle = 0, \ \text{for } i = 1, \ldots, m.$$

with x and y_i, $i = 1, ..., m$, be the optimal primal and dual solutions, respectively.

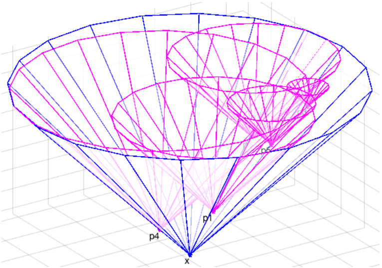
- if $x \prec_{\mathcal{Q}} p_i$ then $y_i = 0$;
- if $y_i \succ_{\mathcal{Q}} 0$ then $x = p_i$ (which can happen at most once);
- if $p_i - x \in \partial\mathcal{Q}$ and $y_i \in \partial\mathcal{Q}$ then $y_{i0}(\overline{p} - \overline{x}) + (p_{i0} - x_0)\overline{y}_i = 0$

$$\Leftrightarrow \quad \overline{y}_i = \frac{y_{i0}}{p_{i0} - x_0} (\overline{x} - \overline{p}_i).$$

Theorem

x is the optimal solution to the primal problem iff $x \preceq_{\mathcal{Q}} p_i$, $i = 1, ..., m$, and

$$\overline{x} \in \operatorname{conv}\left(\overline{p}_i : \|\overline{p}_i - \overline{x}\| = p_{i0} - x_0\right).$$



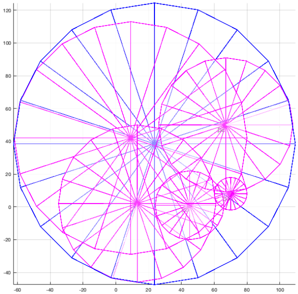


Fig 4. View from the top. The center is in the convex hull of points on the boundary, so it is optimal

The concept of basis

Based on the concept for LP-type problems Matoušek, Sharir & Welzl (1996)

- Let $\mathcal{P}$ be the set of all points, and $\mathcal{P}_1 \subseteq \mathcal{P}$

- Define $w(\mathcal{P}_1) = \mathrm{Inf}_{\mathcal{Q}}(\mathcal{P}_1)$

- A subset $\mathcal{B} \subseteq \mathcal{P}_1$ is a basis if $w(\mathcal{B}') > w(\mathcal{B})$ for all $\mathcal{B}' \subset \mathcal{B}$.
- A basis contains at least 2 points and at most d *affinely independent* points
- $\mathcal{B} \subseteq \mathcal{P}_1$ is a basis for $\mathrm{Inf}_{Q}(\mathcal{P}_1)$ problem if $\mathcal{B}$ is affinely independent, and where the optimal x satisfies $\overline{x} \in \mathrm{ri}\,\mathrm{conv}(\overline{\mathcal{B}})$, with $\overline{\mathcal{B}} = \{\overline{p}_i : p_i \in \mathcal{B}\}$
- The points on a basis $\mathcal{B}$ reside on the boundary $\partial(\mathcal{Q} + x)$

Given a basis, how to find x?

$$\|\overline{p}_i - \overline{x}\|^2 - (p_{i0} - x_0)^2 = \|\overline{p}_1 - \overline{x}\|^2 - (p_{10} - x_0)^2, \quad \forall p_i \in \mathcal{B} \setminus \{p_1\}$$

and

$$\overline{x} \in \operatorname{aff}(\mathcal{B})$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} B^T \\ N^T \end{bmatrix}}_{A} \overline{x} = \underbrace{\begin{pmatrix} b + x_0 c \\ N^T \overline{p}_1 \end{pmatrix}}_{w(x_0)} \quad \text{and} \quad \left\|\overline{p}_1 - A^{-1} w(x_0)\right\|^2 - (p_{10} - x_0)^2 = 0$$

with N a basis for $\mathrm{Null}(\mathrm{Sub}(\overline{B} \cup \{\overline{p}^*\}))$, $B = 2\left[\overline{p}_1 - \overline{p}_1, \ldots, \overline{p}_{|\mathcal{B}|} - \overline{p}_1\right]$,

$$c = 2 \begin{pmatrix} p_{10} - p_{20} \\ \vdots \\ p_{10} - p_{|\mathcal{B}|0} \end{pmatrix} \text{ and } b = \begin{pmatrix} \|\overline{p}_1\|^2 - p_{10}^2 - \|\overline{p}_2\|^2 + p_{20}^2 \\ \vdots \\ \|\overline{p}_1\|^2 - p_{10}^2 - \left\|\overline{p}_{|\mathcal{B}|}\right\|^2 + p_{|\mathcal{B}|0}^2 \end{pmatrix}$$

The dual variables given a basic solution

A basic solution corresponds to a dual feasible solution.

Consider x, the solution to $\mathrm{Inf}_{\mathcal{Q}}(\mathcal{B})$, with $\mathcal{B} \subseteq \mathcal{P}_1$ a basis. We know that:

$$\overline{x} \in \mathrm{conv}(\{\overline{p}_i : p_i \in \mathcal{B}\}) \quad \text{so} \quad \exists\, \alpha_i \geq 0 \text{ s.t. } \overline{x} = \sum_{p_i \in \mathcal{B}} \alpha_i \overline{p}_i, \sum_i \alpha_i = 1,$$

and α_i's are unique. The corresponding dual variables are:

- y_i for $i : p_i \in \mathcal{B}$ is such that:

$$y_{i0} = \frac{\alpha_i (p_{i0} - x_0)}{\sum_j \alpha_j (p_{j0} - x_0)} \quad \text{and} \quad \overline{y}_i = \frac{y_{i0}}{p_{i0} - x_0} (\overline{p}_i - \overline{x}),$$

- $y_i = 0$ for $i : p_i \notin \mathcal{B}$,

which are feasible for the dual problem and satisfy the complementary slackness conditions.

A Dual Simplex Algorithm Based on Dearing and Zeck's dual algorithm (2009)

0. Initialization: It starts with x, the solution $\mathrm{Inf}_{\mathcal{Q}}(\mathcal{B})$ for some basis $\mathcal{B}$ (it is easy to find a basis for a set of two points).

1. Check optimality: If x is primal feasible, then x is the optimal solution to $\mathrm{Inf}_{\mathcal{Q}}(\mathcal{P})$. Else pick p^* primal infeasible.

2. Solve $\mathrm{Inf}_{\mathcal{Q}}(\mathcal{B} \cup \{p^*\})$: Move $\overline{x}$ "towards" the feasibility of p^*, such that the following invariants are maintained:

- The corresponding dual solution is always feasible.
- Complementary slackness is satisfied, that is, the primal constraints corresponding to the basis are binding.

At the end, we have a new basis for the problem $\mathrm{Inf}_{\mathcal{Q}}(\mathcal{B} \cup \{p^*\})$, which is obtained by possibly having to remove some points from the old basis, and by adding p^*. A new iteration then starts

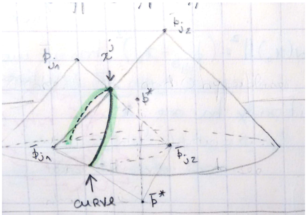
Movement along a curve

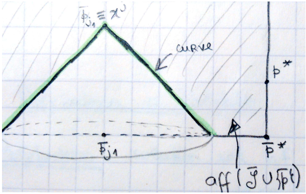
The "curve" is parametrized by t is as follows

- $\|\overline{p}_i - \overline{x}(t)\| = p_{i0} - x_0(t)$ for all $p_i \in \mathcal{B}$
- $\overline{x}(t) \in \operatorname{aff}\left(\mathcal{B} \cup \{\overline{p}^*\}\right)$

And the search is restricted to the polyhedron

$$\mathcal{C} = \left\{ \begin{pmatrix} x_0 \\ \overline{x} \end{pmatrix} \mid \overline{x} \in \operatorname{conv}\left(\mathcal{B} \cup \{\overline{p}^*\}\right) \right\}$$





Two scenarios are possible

- By moving along this curve, we reduce x_0 enough to make p^* become feasible and at $\partial\mathcal{Q}$, and $x_{\text{new}} \in \mathcal{C}$. In this case the pivot is complete and $\mathcal{B}_{\text{new}} = \mathcal{B} \cup \{p^*\}$
- Or before p^* is absorbed into $\mathcal{Q}$, the curve hits the wall of $\mathcal{C}$. In this case one of the points p_i whose dual variable y_i is about to become infeasible must leave the basis:

$$\begin{aligned} \mathcal{B}'_{\text{new}} &\leftarrow \mathcal{B} \setminus \{p_i\} \quad \text{where } y_i = 0 \\ \mathcal{C}_{\text{new}} &\leftarrow \operatorname{conv}\left(\mathcal{B}'_{\text{new}} \cup \{p^*\}\right) \end{aligned}$$

  The curve will now move in the affine space spanned by $\mathcal{C}_{\text{new}}$

  This may have to be repeated several times before p^* becomes feasible (Similar to Goldfarb & Idnani for QP)

The curve $\overline{x}(t)$

$\overline{x}$ moves along the curve $\Delta_{\overline{x}}(t) : \mathbb{R} \to \mathbb{R}^{d-1}$ which has the following properties:

- Primal constraints of $\mathcal{B}$ are binding (complementary slackness is kept):

  $\|\overline{p}_i - (\overline{x} + \Delta_{\overline{x}}(t))\| - p_{i0} = \|\overline{p}_1 - (\overline{x} + \Delta_{\overline{x}}(t))\| - p_{10}, \quad p_i \in \mathcal{B} \setminus \{p_1\}$

$$\Updownarrow$$

$$B^T(\overline{x}(t) + \Delta_{\overline{x}}(t)) = b + x_0(t)c$$
$$\|\overline{p}_1 - (\overline{x}(t) + \Delta_{\overline{x}}(t))\|^2 = (p_{10} - x_0(t))^2$$

- Dual feasibility of $\sum_{i=1}^m y_i(t) = e_0$ is kept:

$$\overline{x} + \Delta_{\overline{x}}(t) \in \mathrm{aff}(\mathcal{B} \cup \{p^*\})$$
$$\Updownarrow$$
$$N^T(\overline{x}(t) + \Delta_{\overline{x}}(t)) = N^T\overline{p}^*$$

  N is a basis for $\mathrm{Null}(\mathrm{Sub}(\overline{B} \cup \{\overline{p}^*\}))$.

- We wish to move towards feasibility of p^*.

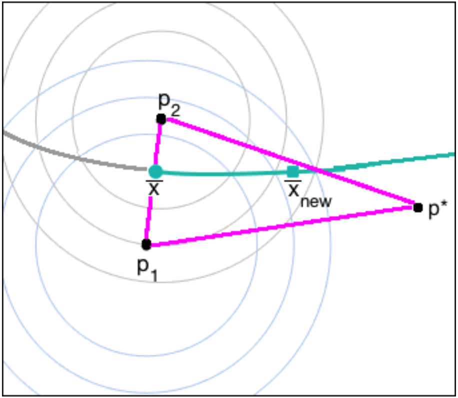


Fig 4. $\Delta_{\overline{x}}(t)$ moving in $\mathcal{C}$.

What if $\Delta_{\overline{x}}(t) = 0$?

This happens when x is the only point such that the primal constraints are binding for the points in $\mathcal{B}$, that is $|\mathcal{B}| = d$.

When this happens, a point needs to be removed from the basis:

- $p_k \in \mathcal{B}$ such that $\overline{x} \in \mathrm{conv}(\{\overline{p}_j : p_j \in \mathcal{B} \setminus \{p_k\} \cup \{p^*\})$

This rule ensures that the dual variables corresponding to x (which are now different from before) are still dual feasible.

The dual variables for $\overline{x} + \Delta_{\overline{x}}(t)$

$\overline{x} + \Delta_{\overline{x}}(t) \in \operatorname{aff}(\overline{\mathcal{B}} \cup \{\overline{p}^*\})$ so $\exists\, \alpha_j$ s.t. $\overline{x} + \Delta_{\overline{x}}(t) = \sum_{p_j \in \mathcal{B} \cup \{p^*\}} \alpha_j \overline{p}_j$, $\sum_{p_j \in \mathcal{B} \cup \{p^*\}} \alpha_j = 1$.

The corresponding dual variables are

$$y_{i0}(t) = \frac{\alpha_i (p_{i0} - x_0(t))}{\sum_{p_j \in \mathcal{B} \cup \{p^*\}} \alpha_j (p_{j0} - x_0(t))}, \quad \overline{y}_i(t) = \frac{y_{i0}}{p_{i0} - x_0}(\overline{p}_i - (\overline{x} + \Delta_{\overline{x}}(t))),\ i : p_i \in \mathcal{B} \cup \{p^*\}$$

$y_i(t) = 0$, $i : p_i \notin \mathcal{B} \cup \{p^*\}$

and these always satisfy $\sum_{i=1}^{m} y_i(t) = e_0$ for all t.

If $\alpha_i < 0$ then $y_i \succ_{\mathcal{Q}} 0$, so y_i becomes dual infeasible. This tells us how far we can move along $\Delta_{\overline{x}}(t)$: until we hit one face of $\operatorname{conv}(\overline{\mathcal{B}} \cup \{\overline{p}^*\})$.

Curve search

We move from $\overline{x}$ along $\Delta_{\overline{x}}(t)$, $t \geq 0$, until the first of the following happens:

1. p^* **becomes primal feasible**: Let x^* be the point on the curve at which this happens. Since $\|\overline{p}^* - \overline{x}^*\| = p_0^* - x_0^*$, to find x^*, we add the following constraint to the set of constraints that define any point on the curve:

$$2[p^* - p_1]^T \overline{x}^* = 2x_0^* [p_{10} - p_0^*] + \left[\|\overline{p}_1\|^2 - p_{10}^2 - \|\overline{p}^*\|^2 + (p_0^*)^2 \right]$$

2. **a face of** $\mathrm{conv}(\overline{\mathcal{B}} \cup \{\overline{p}^*\})$ **is hit**: Let x_i be the point s.t. $\overline{x}_i$ is the intersection of the curve with F_i, the face opposed to $\overline{p}_i \in \mathcal{B}$. To find it we get N_i, a basis of $\mathrm{Null}(\mathrm{Sub}(\overline{B} \setminus \{\overline{p}_i\} \cup \{\overline{p}^*\}))$:

$$N_i^T \overline{x}_i = N_i^T \overline{p}^*$$

   Calculate x_i for every face, and select the one with maximum x_{i0} s.t. $\langle \overline{p}^* - \overline{x}, \overline{x}_i - \overline{x} \rangle > 0$ (the direction improving feasibility of p^*).

Updating the basis after the curve search

The case that happens first is the one whose corresponding point has the largest height.

1. **When p^* becomes feasible first**: The new solution is now defined by a new basis $\mathcal{B} = \mathcal{B}' \cup \{\mathrm{p}^*\}$. And, we start a new iteration.

2. **When a face of $\mathrm{conv}(\overline{\mathcal{B}} \cup \{\overline{\mathrm{p}}^*\})$ is hit first**:

   - The solution of $\mathrm{Inf}_{\mathcal{Q}}(\mathcal{B} \cup \{\mathrm{p}^*\})$ is not defined by the corresponding p_i, therefore it is removed from the basis $\mathcal{B} = \mathcal{B} \setminus \{\mathrm{p}_i\}$.
   - We go back to finding a new curve now with the new basis.

Theorem

At each iteration the objective function value, x_0, strictly decreases, and since it stops when all points are covered, the algorithm is finite.

Efficiency of the pivot

- When $\mathcal{B}_{\text{new}} = \mathcal{B} \cup \{p^*\}$, that is no wall of $\mathcal{C}$ was hit, then the new basis and the new x can be obtained by a rank-one update of the previous system computing the old x
- When a wall of $\mathcal{C}$ is hit a point in $\mathcal{B}$ has to be dropped, the new x can be computed by rank-one update of the previous system
- Every time a wall is hit and another rank-one update must be solved
- By maintaining a QR factorization rank-one updates can be achieved efficiently ($\mathcal{O}(d^2)$)

Extensions

- We may replace $\mathcal{Q}$ in principle with any *proper cone* $\mathcal{K}$ and seek $\mathrm{Inf}_{\mathcal{K}}$, these are, in principle LP-type problems
- Of particular interest is the cone of nonnegative univariate polynomials over an interval $[a, b]$
- Use the dual algorithm to solve the problem of partial enclosure (when only a fraction of the given points are to be covered).
- Another set of LP-type problems: Minimum volume ellipsoid containing a set of points, or a set of ellipsoids