The Ascendance of the **Dual** Simplex Method: A Geometric View

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The Ascendance of the Dual Simplex Method: A Geometric View

First described in the 1950s, the dual simplex evolved in the 1990s to become the method most often used in solving linear programs. Factors in the ascendance of the dual simplex method include Don Goldfarb’s proposal for a steepest-edge variant, and an improved understanding of the bounded-variable extension. The ways that these come together to produce a highly effective algorithm are still not widely appreciated, however. This talk employs a geometric approach to the dual simplex method to provide a unified and straightforward description of the factors that work in its favor.
Motivation

Progress in the dual simplex method for large scale LP problems: practical dual phase 1 algorithms

Achim Koberstein - Uwe H. Suhl

1 Introduction

Lemke [18] developed the dual simplex method in 1954 but it was not found to be an alternative to the primal simplex method for nearly forty years. This changed in the early Nineties mainly due to the contributions of Forrest and Goldfarb [7] and Fourer [8]. During the last decade commercial solvers have made great progress in

Primal Linear Program

Minimize \( cx \)
Subject to \( Ax = b, \ x \geq 0 \)

**Basic variables** \( B \), **nonbasic variables** \( N \)
- Coefficient columns of basic variables form a nonsingular matrix \( B \)

**Basic solution**
- \( \bar{x}_N = 0, B\bar{x}_B = b \)

**Feasible basic solution**
- \( \bar{x}_B \geq 0 \)
Dual Linear Program

Maximize $\pi b$
Subject to $\pi A \leq c$

**Binding constraints $B$, nonbinding constraints $N$**

- Coefficient rows of binding constraints form a nonsingular matrix $B$

**Vertex solution**

- $\bar{\pi}B = c_B$

**Feasible vertex solution**

- $\bar{\pi}A_N \leq c_N$
Dual Linear Program

Maximize \( \pi b \)
Subject to \( \pi A + \sigma = c, \ \sigma \geq 0 \)

**Binding constraints** \( B \), **nonbinding constraints** \( N \)

- Coefficient rows of binding constraints form a nonsingular matrix \( B \)

**Vertex solution**

- \( \bar{\sigma}_B = 0, \bar{\pi}B = c_B \)

**Feasible vertex solution**

- \( \bar{\sigma}_N = c_N - \bar{\pi}A_N \geq 0 \)
Primal Simplex Method

*Given*
- feasible basic solution $\bar{x}$ and corresponding basis matrix $B$

*Choose a nonbasic variable to enter*
- solve $\pi B = c_B$
- select $p \in N: \sigma_p = c_p - \pi a_p < 0$

*Choose a basic variable to leave*
- solve $B y_B = a_p$
- select $q \in B: \Theta = \bar{x}_q / y_q = \min_{y_i > 0} \bar{x}_i / y_i$

*Update*
- $\bar{x}_p \leftarrow \Theta$
- $\bar{x}_i \leftarrow \bar{x}_i - \Theta y_i$ for all $i \in B$
Dual Simplex Method

Given

- feasible vertex solution $\bar{\sigma}$ and corresponding basis matrix $B$

Choose a binding constraint to leave

- solve $Bx_B = b$
- select $q \in B$: $x_q < 0$

Choose a nonbinding constraint to enter

- solve $\delta B = e_q$
- select $p \in N$: $\Phi = \bar{\sigma}_p/\delta a_p = \min_{\delta a_j > 0} \bar{\sigma}_j/\delta a_j$

Update

- $\bar{\sigma}_q \leftarrow \Phi$
- $\bar{\sigma}_j \leftarrow \bar{\sigma}_j - \Phi(\delta a_j)$ for all $j \in N$
Inner Products with \(a_j\): Column-Wise

*Work of \(v \cdot a_j\) is the same for any \(v\)*

- For each nonzero \(a_{ji}\) in column \(j\) of \(A\), add \(v_i a_{ji}\) to sum
- Need all \(v_i\) in an \(m\)-vector

**Primal simplex**

- Select one \(\sigma_p = c_p - \pi a_p < 0\) (\(\pi B = c_B\)) for \(p \in \mathcal{N}\)
- \(\leq |\mathcal{N}|\) inner products, but often \(\ll |\mathcal{N}|\)

**Dual simplex**

- Compute \(\min_{\delta q a_j > 0} \bar{\sigma}_j / \delta q a_j\) (\(\delta q B = e_q\))
- Always \(|\mathcal{N}|\) inner products
Inner Products with $a_j$: Row-Wise

*Store $A$ by row as well as by column*
  - Accumulate $n$ inner products together

*Work of $v \cdot A$ depends on sparsity of $v$*
  - For each nonzero $v_i$,
    - for each nonzero $a_{ji}$ in row $i$ of $A$, add $v_i a_{ji}$ to sum for $a_j$

*Faster $\delta_q a_j$ in dual simplex*
  - $\delta_q = e_q B^{-1}$ tends to be especially sparse

*Faster $\sigma_j = c_j - \pi a_j$ in primal if you update them all*
  - $\sigma_q \leftarrow \sigma_p / \delta_q a_p$
  - $\sigma_j \leftarrow \sigma_j - \sigma_q (\delta_q a_j)$ for all $j \in \mathcal{N}$
Primal Steepest Edge

North-Holland Publishing Company

A PRACTICABLE STEEPEST-EDGE SIMPLEX ALGORITHM

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It is shown that suitable recurrences may be used in order to implement in practice the steepest-edge simplex linear programming algorithm. In this algorithm each iteration is along an edge of the polytope of feasible solutions on which the objective function decreases most rapidly with respect to distance in the space of all the variables. Results of computer comparisons on medium-scale problems indicate that the resulting algorithm requires less iterations but about the same overall time as the algorithm of Harris [8], which may be regarded as approximating the steepest-edge algorithm. Both show a worthwhile advantage over the standard algorithm.

Key words: Simplex Method, Linear Programming, Steepest-edge, LU Factorization.
Primal Steepest Edge

**Simplex step**
- If $x_j$ enters, solution $\bar{x}$ changes by $\theta y_j$
  - Entering variable $\bar{x}_p$ increases to $\theta$
  - Basic variables $\bar{x}_B$ change by $\theta y_B$ (where $By_B = a_p$)
- Objective is reduced by $\theta \sigma_j$

**Steepness of step**
- $\frac{\sigma_j}{\|y_j\|} = $ reduction of objective per unit change in solution

**Main steepest-edge computations**
- Choose largest $\sigma_j^2 / y_j^T y_j$ over all $j \in \mathcal{N}$ with $\sigma_j < 0$
- Update $\sigma_j$ for $j \in \mathcal{N}$
- Update $y_j^T y_j$ for $j \in \mathcal{N} \ldots$
Primal Steepest Edge

**Updating** $y_j^T y_j$

- $y_j \leftarrow y_j - \alpha_j y_p$ ($\alpha_j = y_{jq}/y_{pq}$)
  - $\alpha_j$ known after updating $\sigma_j$
  - $y_j$ not known except for $y_p$, but $y_j^T y_j$ is known

- $y_j^T y_j \leftarrow (y_j - \alpha_j y_p)^T (y_j - \alpha_j y_p)$
- $y_j^T y_j \leftarrow y_j^T y_j - 2\alpha_j y_j^T y_p + \alpha^2 y_p^T y_p$

**Hard part is** $y_j^T y_p = a_j^T B^{-T} y_p$

- Solve $wB = y_p$
- Then compute $y_j^T y_p$ as $a_j^T w$ for each $j \in \mathcal{N}$
- One extra solve and $|\mathcal{N}|$ extra inner products
Dual Steepest Edge

Sparse Matrix Computations

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Dual Steepest Edge

Using the Steepest-Edge Simplex Algorithm to Solve Sparse Linear Programs.

By D. Goldfarb
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Sometimes the solutions of the LP problem (1.1)-(1.3) are required for several different vectors b in (1.2). In such a situation or when a constraint is added to an LP problem whose solution is already known, it is advantageous to use a dual feasible algorithm. Thus we consider briefly a dual steepest-edge simplex algorithm. One might better call such an algorithm a maximal distance algorithm since at each step the pivot row selected is the one which in the transformed set of equations

\[ B^{-1}A \mathbf{x} = B^{-1}b, \]

has a negative right hand side and whose corresponding hyperplane is furthest from the origin. Algebraically, one considers the elements of \( B^{-1}b \) weighted by the norms of the corresponding rows.

Under the assumptions of section 2 it is simple to show that the following recurrences hold for the rows of \( B^{-1}A \), \( \rho_i = \mathbf{e}_i^T B^{-1}A \), and the square of their norms \( \beta_i = \rho_i^2 \rho_i \).
Dual Steepest Edge

\[
\bar{p}_q = \frac{p_p}{w_p} \quad (5.5a)
\]

\[
\bar{p}_i = \frac{p_i - \omega_i \bar{p}_p}{w_i} \quad i \leq m, i \neq p \quad (5.5b)
\]

and

\[
\bar{\beta}_q = \frac{\beta_p}{w_p^2} \quad (5.6a)
\]

\[
\bar{\beta}_i = \beta_i - 2(w_i/w_p)\alpha_i T_B^{-1} A^T B^{-1} \alpha_i \quad i \neq p \quad (5.6b)
\]

where \( A = [B; A] \).

As in the primal algorithm the pivot column \( w = B^{-1} a_q \) and all elements of the pivot row \( \alpha_i = a_i T_B^{-1} a_i \) must be computed. Note that \( \bar{\beta}_1 \geq 1 + w_1^2/w_p^2 \). In the primal algorithm \( B^{-T} w \) is needed whereas here we require \( B^{-1} y \) where \( y = E a_1 \). \( y \) can be computed in the same loop as the \( \alpha_i \)'s. As in the primal case most of these will be zero in sparse problems with a consequent reduction in the work required to compute \( y \). Small savings also result from zeros in the pivot column.

The vector of row weights \( \beta \) can also be initialized economically by setting all of its components to one and then adding to these the square of the respective components of \( B^{-1} a_i \) for \( j > m \). Thus it is clear that it is possible to implement a practicable dual steepest-edge algorithm for large sparse LP problems.
Dual Steepest Edge

Simplex step
- If constraint $i \in \mathcal{B}$ is relaxed,
  - Slack variable $\sigma_i$ increases to $\phi$
  - Variables $\pi$ change by $\phi \delta_i$ (where $\delta_i B = e_i$)
- Objective is reduced by $\phi x_i$

Steepness of step
- $x_i / \|\delta_i\| =$
  reduction of objective per unit change in solution

Main steepest-edge computations
- Choose largest $x_i^2 / \delta_i^T \delta_i$ over all $i \in \mathcal{B}$ with $x_i < 0$
- Update $x_i$ for $i \in \mathcal{B}$
- Update $\delta_i^T \delta_i$ for $i \in \mathcal{B}$ . . .
**Dual Steepest Edge**

**Updating $\delta_i^T \delta_i$**

- $\delta_i \leftarrow \delta_i - \beta_i \delta_q$ ($\beta_i = y_{pi}/y_{pq}$)
  - $\beta_i$ known after updating $x_i$
  - $\delta_i$ not known except for $\delta_q$, but $\delta_i^T \delta_i$ is known

- $\delta_i^T \delta_i \leftarrow (\delta_i - \beta_i \delta_q)^T (\delta_i - \beta_i \delta_q)$
- $\delta_i^T \delta_i \leftarrow \delta_i^T \delta_i - 2\beta \delta_j^T \delta_i \delta_q + \beta^2 \delta_q^T \delta_q$

**Hard part is $\delta_i^T \delta_q = e_i B^{-1} \delta_q$**

- Solve $B v = \delta_q$
- Then $\delta_i^T \delta_q$ is $v_i$ for each $i \in B$
- One extra solve but *no extra inner products*
Bounded Variables

Notes on the Dual Simplex Method

0. The standard dual simplex method
1. A more general and practical dual simplex method
2. Phase I for the dual simplex method
3. Degeneracy in the dual simplex method
4. A generalized ratio test for the dual simplex method

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Bounded Variables

*Generalize* $x \geq 0$ *to* $\ell \leq x \leq u$

- State simplex methods for $\ell, u$ finite
- Extend to allow some $\ell_j = -\infty$ and/or $u_j = +\infty$
- Check that $\ell = 0, u = \infty$ reduces to previous case

*Further improve the dual simplex method*

- Take longer steps
- Adapt to degeneracy
Primal LP with Bounded Variables

\begin{align*}
\text{Minimize} & \quad cx \\
\text{Subject to} & \quad Ax = b, \ \ell \leq x \leq u
\end{align*}

Basic variables $\mathcal{B}$, nonbasic variables $\mathcal{N} = \mathcal{L} \cup \mathcal{U}$

- Coefficient columns of basic variables form a nonsingular matrix $B$

Basic solution

- $\bar{x}_j = \ell_j$ for $j \in \mathcal{L}$, $\bar{x}_j = u_j$ for $j \in \mathcal{U}$
- $B \bar{x}_B = b - \sum_{j \in \mathcal{L}} \ell_j a_j - \sum_{j \in \mathcal{U}} u_j a_j$

Feasible basic solution

- $\ell_B \leq \bar{x}_B \leq u_B$
Dual LP with Bounded Variables

Nonlinear Programming 4

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Dual LP with Bounded Variables

A Second Order Method for the Linearly Constrained Nonlinear Programming Problem
Garth P. McCormick
University of Wisconsin, Madison, Wisconsin

Convergent Step-Sizes for Gradient-Like Feasible Direction Algorithms for Constrained Optimization
James W. Daniel
University of Wisconsin, Madison, Wisconsin

On the Implementation of Conceptual Algorithms
E. Polak
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Recent Results on Complementarity Problems
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Dual LP with Bounded Variables

Obviously there is no difficulty in passing from $f_j$ to $g_j$ to $p_j$ by the method above when $p_j$ is of such type.

To drive this point home and make apparent the directness and flexibility of this duality scheme in monotropic programming, we turn to the example of (P) as a general linear programming problem with both upper and lower bounds for each variable. Linear programming theory is incapable of producing a dual without first subjecting the problem to a transformation into one of the canonical forms where no single variable is bounded in both directions. In the monotropic programming context, however, we can regard a problem of this sort as

Note that in this example the dual of a linear programming problem turns out, in general, to be merely piecewise linear. It is no wonder, then, that linear programming theory cannot fully capture such duality. Other forms of linear programming problems can be handled similarly. In essence, one
**Dual LP with Bounded Variables**

Maximize \( \pi b + [u: \ell] \sigma \)

Subject to \( \pi A + \sigma = c \)

**What is \([u: \ell] \sigma\)?**

- Sum of concave piecewise-linear functions \([u_j: \ell_j] \sigma_j\)
  - Slope of \(u_j\) for \(\sigma_j \leq 0\)
  - Slope of \(\ell_j\) for \(\sigma_j \geq 0\)
- Example for \(0 \leq \ell_j < u_j < \infty\):

![Diagram illustrating the concept of \([u: \ell] \sigma\) with \(u_j\), \(\ell_j\), and \(\sigma_j\) axes.](image)
Dual LP with Bounded Variables

Maximize $\pi b + [u: \ell] \sigma$

Subject to $\pi A + \sigma = c$

**Binding constraints** $B$,
**nonbinding constraints** $N = L \cup U$

- Coefficient rows of binding constraints form a nonsingular matrix $B$

**“Vertex” solution**
- $\bar{\sigma}_B = 0, \bar{\pi}B = c_B$
- $j \in L$ for $\bar{\sigma}_j = c_j - \bar{\pi}a_j > 0$
- $j \in U$ for $\bar{\sigma}_j = c_j - \bar{\pi}a_j < 0$

*Always feasible!*
Primal Simplex, Bounded Variables

Given

- feasible basic solution $\bar{x}$ and corresponding basis matrix $B$

Choose a nonbasic variable to enter

- solve $\pi B = c_B$
- select $p \in \mathcal{L}$: $\sigma_p = c_p - \pi a_p < 0$ or
- select $p \in \mathcal{U}$: $\sigma_p = c_p - \pi a_p > 0$

Choose a basic variable to leave

- solve $By_B = a_p$ ($p \in \mathcal{L}$) or $By_B = -a_p$ ($p \in \mathcal{U}$)
- select $\Theta = \min (\Theta_\mathcal{L}, \Theta_\mathcal{U}, \Theta_p)$:
  - $q \in \mathcal{B}$: $\Theta_\mathcal{L} = (\bar{x}_q - \ell_q)/y_q = \min_{y_i>0} (\bar{x}_i - \ell_i)/y_i$
  - $q \in \mathcal{B}$: $\Theta_\mathcal{U} = (u_q - \bar{x}_q)/y_q = \min_{y_i<0} (\bar{x}_i - u_i)/y_i$
  - $q = p$: $\Theta_p = u_p - \ell_p$

Update...
Dual Simplex, Bounded Variables

Given

- feasible vertex solution $\bar{\sigma}$ and corresponding basis matrix $B$

Choose a binding constraint to leave

- solve $Bx_B = b - \sum_{j \in L} \ell_j a_j - \sum_{j \in U} u_j a_j$
- select $q \in B$: $x_q < \ell_q$ or $x_q > u_q$

Choose a nonbinding constraint to enter

- solve $\delta B = e_q$ (if $x_q < \ell_q$) or $\delta B = -e_q$ (if $x_q > u_q$)
- select $\Phi = \min (\Phi_L, \Phi_U)$:
  * $j \in L$: $\Phi_L = \bar{\sigma}_p / \delta a_p = \min_{\delta a_j > 0} \bar{\sigma}_j / \delta a_j$
  * $j \in U$: $\Phi_U = \bar{\sigma}_p / \delta a_p = \min_{\delta a_j < 0} \bar{\sigma}_j / \delta a_j$

Update . . .
In Principle, Bounds Can Be Infinite

Some $\ell_j = -\infty$ and/or $u_j = +\infty$ ??

- A basis may be infeasible
  * $\bar{\sigma}_j > 0$ but $\ell_j = -\infty$
  * $\bar{\sigma}_j < 0$ but $u_j = +\infty$

- Minimize “sum of infeasible variables” to get feasible
  * Replace each $\ell_j = -\infty$ by $-1$
  * Replace each $u_j = +\infty$ by $+1$
  * Replace all finite bounds by $0$

All $\ell_j = 0$ and $u_j = +\infty$ ??

- Then $x_j \geq 0$
- Primal and dual algorithms reduce to their simpler forms
In Practice, Bounds Tend to be Finite

**Decisions are bounded**
- Variables are bounded
- Slacks on inequality constraints are bounded

**Integer-valued decisions have small values**
- Many are zero-one!

**Standard presolve routines compute bounds**
- Compute bounds where none given by user
- Tighten bounds in multiple passes
Long Steps

Standard iteration

- $\bar{\sigma}_q$ increases to $\Phi$ (if $x_q < \ell_q$) or decreases to $-\Phi$ (if $x_q > u_q$)
- $\bar{\sigma}_p$ moves to 0

Suppose $\bar{\sigma}_q$ increases/decreases further . . .

- $\bar{\sigma}_p$ moves past 0, but solution remains feasible
- Rate of improvement in objective degrades by $|\delta a_p| (u_p - \ell_p)$
  - If $\bar{\sigma}_p$ increasing, objective slope changes from $u_p$ to $\ell_p$
  - If $\bar{\sigma}_p$ decreasing, objective slope changes from $-\ell_p$ to $-u_p$
- If rate still positive, can continue until next $\bar{\sigma}_j$, $j \in \mathcal{N}$ reaches 0
Long Step Iteration

Replace single ratio test by a loop

- Set initial objective improvement rate to
  \[ r = \ell_q - x_q \text{ (if } x_q < \ell_q \text{) or } r = x_q - u_q \text{ (if } x_q > u_q \text{)} \]
- Form set of all ratios that may be reached:
  \[ Q = \{ \bar{\sigma}_j / \delta a_j : j \in \mathcal{L}, \delta a_j > 0 \} \cup \{ \bar{\sigma}_j / \delta a_j : j \in \mathcal{U}, \delta a_j < 0 \} \]
- Repeat for increasing \( \bar{\sigma}_j / \delta a_j \in Q : \)
  * Let \( r \leftarrow r - |\delta a_j|(u_j - \ell_j) \)
  * Let \( Q \leftarrow Q \setminus \{ \bar{\sigma}_j / \delta a_j \} \)

until \( r \leq 0 \)

Equivalent to “weighted selection”

- In theory, faster than sorting
- In practice, a small part of iteration cost
Re-Optimization for MIPs

*Change bounds for some fractional* $\bar{x}_i$, $i \in B$

- Increase $\ell_i$ to $[\bar{x}_i]$, resulting in $\bar{x}_i < \ell_i$
- Decrease $u_i$ to $[\bar{x}_i]$, resulting in $\bar{x}_i > u_i$
- Either way, binding constraint $i \in B$ can leave
- Continue with dual simplex steps until optimal again

*Fix some fractional binary* $\bar{x}_i$, $i \in B$

- Increase $\ell_i$ to $u_i = 1$, resulting in $\bar{x}_i < \ell_i$
- Decrease $u_i$ to $\ell_i = 0$, resulting in $\bar{x}_i > u_i$
- Either way, binding constraint $i \in B$ can leave
- Since now $\ell_i = u_i$, using long steps
  the constraint will never become binding again
Degeneracy

Choosing a binding constraint $q$ to relax

- For $\bar{\sigma}_j > 0$, place $j \in \mathcal{L}$
- For $\bar{\sigma}_j < 0$, place $j \in \mathcal{U}$
- For $\bar{\sigma}_j = 0$ ???
  - * Guess $j \in \mathcal{L}$ if you think $\bar{\sigma}_j$ is likely to increase
  - * Guess $j \in \mathcal{U}$ if you think $\bar{\sigma}_j$ is likely to decrease
    
    ... can’t be sure though until you choose $q$

- solve $Bx_B = b - \sum_{j \in \mathcal{L}} \ell_j a_j - \sum_{j \in \mathcal{U}} u_j a_j$
- select $q \in \mathcal{B}$: $x_q < \ell_q$ or $x_q > u_q$
### Degeneracy

**Choosing a nonbinding constraint \( p \) to add**

- Set initial objective improvement rate to
  \[
  r = \ell_q - x_q \quad \text{(if} \ x_q < \ell_q \text{)} \quad \text{or} \quad r = x_q - u_q \quad \text{(if} \ x_q > u_q \text{)}
  \]
- Collect all ratios that may be reached:
  \[
  Q = \left[ \frac{\bar{\sigma}_j}{\delta a_j}: j \in \mathcal{L}, \delta a_j > 0 \right] \cup \left[ \frac{\bar{\sigma}_j}{\delta a_j}: j \in \mathcal{U}, \delta a_j < 0 \right]
  \]
- Some of these ratios may be zero!
  - For \( j \in \mathcal{L}, \bar{\sigma}_j = 0 \) and \( \delta a_j > 0 \)
  - For \( j \in \mathcal{U}, \bar{\sigma}_j = 0 \) and \( \delta a_j < 0 \)
- Repeat for every \( \frac{\bar{\sigma}_j}{\delta a_j} = 0 \in Q \):
  - Let \( r \leftarrow r - \left| \delta a_j \right|(u_j - \ell_j) \)
  - Let \( Q \leftarrow Q \setminus \{\bar{\sigma}_j/\delta a_j\} \)
  while \( r > 0 \)
- If \( r \leq 0 \), iteration is degenerate
- If still \( r > 0 \), continue with nondegenerate iteration
Degeneracy: Benefit of Long Steps

For some $\bar{\sigma}_j = 0$, you “guess wrong”

- $j \in \mathcal{L}$, but $\bar{\sigma}_j$ decreases
- $j \in \mathcal{U}$, but $\bar{\sigma}_j$ increases

As a result, you are overly optimistic about the objective improvement rate $r$

Long-step ratio test corrects for your wrong guesses

- If the corrected $r > 0$ then you can take a nondegenerate step after all

None of this changes the optimality condition

- $\ell_q \leq x_q \leq u_q$ for all $q \in \mathcal{B}$
Ascendance of the Dual Simplex

Fast inner products

Dual steepest edge

Bounded-variable extension

- Feasibility for finite bounds
- Long steps
- More nondegenerate steps