Optimization approaches to combat Gerrymandering

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Historical Presidential Election in Wisconsin

![Graph showing raw votes by year for DEM and REP parties in Wisconsin elections from 1996 to 2020. The x-axis represents the years 1996 to 2020, and the y-axis represents raw votes. The graph indicates a general increase in raw votes over the years, with a peak in 2012 and 2020.](image-url)
Historical US House Election in Wisconsin

![Bar Chart](image)

- **Democrats**
- **Republicans**

Year:
- 2002
- 2004
- 2006
- 2008
- 2010
- 2012
- 2014
- 2016
- 2018
- 2020
- 2022

Number of Votes:
- 0.0
- 0.2
- 0.4
- 0.6
- 0.8
- 1.0
- 1.2
- 1.4
- 1.6
Ideal Seat assignment vs. real Seat assignment

Ideal US House Seats for Republicans = round\((8 \times \frac{\text{Republican raw votes}}{\text{total raw votes}})\)

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Map change from 2008 to 2010
Finding what works

- For simple models, can be solved by optimization methods
- For games, everything rigid, can solve by brute force search
- For more complex models, can solve by DFO
  \textit{this is what we do in randomized controlled trials}
- To understand people, what happens when the distribution changes?, when the moves change their meaning?, when the desired outcome changes?, when the outcome is unclear?, when outcomes are not comparable?…
The rules

- Districts shall comply with the United States Constitution, the Voting Rights Act of 1965, and all applicable federal laws;
- Districts shall be drawn on the basis of inhabitants;
- Districts shall be geographically contiguous;
- Districts shall provide racial and language minorities with an equal opportunity to participate in the political process;
- Districts shall respect the integrity of communities of interest to the extent practicable.
- Districts shall not split precincts and shall respect the geographic integrity of political subdivision boundaries to the extent that preceding criteria have been satisfied.
- The redistricting plan shall not, when considered on a statewide basis, unduly favor or disfavor any political party.
Data used in our application

For our applications, we used ward-level district data. The state of Wisconsin is divided into 7078 wards in total.

1. Wisconsin shape data that contains the coordinated for each ward polygon
2. Wisconsin US House voting data for each ward
3. Wisconsin racial data for each ward

Wisconsin wards
Problem definition (rule interpretation)

- How do we divide this map into 8 congressional districts?
- Redraw map boundaries to do this fairly.

1. Assign each ward to a district
2. Satisfying Contiguous Constraint: a district should not be broken into multiple parts.
3. Satisfying Population Constraint: the population of every district should satisfy some predefined bounds.
4. Trying to Maximize the Compactness of Districts: prefer districts with more compact shape instead of long and thin districts.
Problem definition

Assume each grid is a ward and each unique color means an assignment to a district. The strategy on the left is more compact than the one on the right.

In the above scenario, the strategy does not satisfy contiguous constraint since the red district is separated.
Methodology of Gerrymandering

Objective:

$$\min \sum_{i \in V} \sum_{j \in V} w_{ij} x_{ij}$$  \hspace{1cm} (1a)$$

Constraints:

$$\sum_{j \in V} x_{ij} = 1 \hspace{1cm} \forall i \in V$$  \hspace{1cm} (1b)$$

$$\sum_{j \in V} x_{jj} = k$$  \hspace{1cm} (1c)$$

$$L x_{jj} \leq \sum_{i \in V} p_i x_{ij} \leq U x_{jj} \hspace{1cm} \forall j \in V$$  \hspace{1cm} (1d)$$

$$x_{ij} \leq x_{jj} \hspace{1cm} \forall i, j \in V$$  \hspace{1cm} (1e)$$

$$x_{ij} \in \{0, 1\} \hspace{1cm} \forall i, j \in V$$  \hspace{1cm} (1f)$$

contiguity constraints  \hspace{1cm} (1g)$$
Objective function for the optimal MIP model. Suppose there are two districts. One in blue and the other in yellow, and the district centers are units that contain the red centroid point. The objective value is the sum of the distance between each unit and its assigned district center’s centroid; i.e., the sum of thin black lines length in the figure.
Methodology of Gerrymandering - The SHIR Model

We can represent the shape data as a node edge graph with each node representing a ward and each edge representing the adjacency of two wards.

The SHIR model uses network flow constraints to maintain contiguity.
Suppose D is the district center, then ward A can be assigned to district center D only if one of its adjacent wards that is closer to D (which is B in the above picture) is assigned to D.
Sampling compact maps - generation algorithm

Start
Input:
1. Population bound
2. District Number
3. Population ratio

Generate initial map by MIP optimization of compactness measure.
Greedy improvement algorithm: select adjacent maximal population disparities and reassign enforcing compactness and contiguity metrics.

Is population balanced?
Yes
No

End
Fair maps are not “fair”

The bar chart on the left is the distribution of Democrat congressional Seats in 2020 US House election. We’ve generated 300 hundred maps based on the population of 2020 presidential elections. Note the ideal congressional seats for Democrats are 4.
Policy change: multi-representative districts

<table>
<thead>
<tr>
<th>District number</th>
<th>Population ratio</th>
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<td>5</td>
<td>2:2:2:1:1</td>
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<tr>
<td>6</td>
<td>2:2:1:1:1:1</td>
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<tr>
<td>7</td>
<td>2:1:1:1:1:1:1</td>
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<tr>
<td>8</td>
<td>1:1:1:1:1:1:1:1</td>
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Voting Calculation Policy

**Winner takes all**
All congressional Seats are assigned to the winning party in a district.

**Assigned by weights**
Congressional Seats are assigned to the parties according to the voting ratio in a district.

We generate 300 maps for every district number, and then do experiments on the two allocation policies.
Examples of multi-representative maps

5 districts

6 districts

7 districts

8 districts
Democratic Seats Won: winner-takes-all (2020 UHS)

<table>
<thead>
<tr>
<th>number of districts</th>
<th>1 seat</th>
<th>2 seats</th>
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<td>4</td>
<td>7</td>
<td>73</td>
<td>197</td>
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<td>116</td>
<td>155</td>
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Democrats Seats Won: proportional (2020 UHS)

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Use Stochastic Programming to Counter Adversary

A general stochastic framework can be represented in the following form:

\[
\begin{align*}
\text{Cost} & \quad \text{Risk} \\
\min & \quad c(x) + \lambda t + (1 - \lambda)\left\{ \sum_{i=1}^{n} y_i + \frac{1}{1 - \alpha} \sum_{s \in S} P(s)u_{is} \right\} \\
\text{s.t.} & \quad \sum_{i=1}^{n} x_i = 1 \\
& \quad t \geq x_i \sum_{s \in S} P(s)\text{Deviation}(i, s) \quad \forall i = 1, \ldots, n \\
& \quad u_{is} \geq \text{Deviation}(i, s)x_i - y_i \quad \forall s \in S, i = 1, \ldots, n \\
& \quad x \in \{0, 1\}^n
\end{align*}
\]

<table>
<thead>
<tr>
<th>Cost Term</th>
<th>Risk Term</th>
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<tr>
<td>1. Compactness Measures</td>
<td>1. Deviation Measures (Fairness)</td>
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<td>2. Differences between the current map and the proposed map</td>
<td>2. Population Imbalance</td>
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Objective over scenarios
### 8 district setting

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**Chosen Map**

[Map of Wisconsin]

Ferris, Lu, Luedtke (UW-Madison)  
Gerry for WI  
US/Mexico Opt
SP Selected Map (6 Districts)
## Countering adversaries

### Winner-take-all

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Conclusion and Outlook

- Fair maps are not necessarily “fair”
- Adversarial approach (policy and demographic changes) leads to fairer solutions
- Forward versus backward model
- Causality, visualization, clarity of model is key for policy impact
- Currently interacting with state representatives to see how to further discussion.