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Revelation Gap in Prior-independent Mechanism Design

Yiding Feng

Abstract

The focus of this thesis is to provide a theoretical understanding of the potential inadequacy of revelation principle and advantages of non-truthful mechanisms. We propose revelation gap – a quantification of optimal prior-independent approximation ratio among all revelation mechanisms vs. the optimal prior-independent approximation ratio among all (possibly non-revelation) mechanisms. We prove the existence of non-trivial revelation gaps in canonical environments.

Keywords

algorithmic game theory, mechanism design, revenue maximization, welfare maximization, revelation gap

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Revelation Gap in Prior-independent Mechanism Design

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ABSTRACT

Revelation Gap in Prior-independent Mechanism Design

Yiding Feng

Motivated by real-world problems in various fields, mechanism design governs the design of protocols for strategic agents and has applications both in computer science and economics. Due to the revelation principle – a seminal observation in mechanism design, a vast number of studies in mechanism design focus on revelation mechanisms (i.e., ones where revealing preferences truthfully forms an equilibrium). However, successful applications (e.g., first-price auction, generalized second-price auction for advertisers in sponsored search) suggest a great practical impact for non-revelation mechanisms.

The focus of this thesis is to provide a theoretical understanding of the potential inadequacy of revelation principle and advantages of non-truthful mechanisms. We consider questions from prior-independent mechanism design, namely identifying a single mechanism that has near optimal performance on every prior distribution of agents' preferences. To characterize the loss of the restriction to revelation mechanisms, we propose revelation gap – a quantification of optimal prior-independent approximation ratio among all revelation mechanisms vs. the optimal prior-independent approximation ratio among all (possibly non-revelation) mechanisms. We prove the existence of non-trivial revelation

gaps in two canonical environments: (i) welfare maximization for public budgeted agents, and (ii) revenue maximization for a linear agent with a single sample access.

Our analysis methods are of broader interest in mechanism design, and the study suggests that it is important to systematically develop a theory for the design of non-revelation mechanisms.

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Dedication

To my family.

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CHAPTER 1

Introduction

Mechanism design studies how to design protocols (i.e., mechanisms) to achieve desired outcomes (e.g., resource allocations) for strategic agents who optimize their action based on their own preference. This line of research is motivated by real-world problems in various fields, and leads to a vast number of practical applications (e.g., spectrum auctions – Bichler and Goeree, 2017, monopoly regulation – Baron and Myerson, 1982, environmental policy – Cramton and Kerr, 2002). A more recent example is the auctions for Internet advertising, where different auction formats (e.g., first-price auction, second price auction, generalized second-price auction) studied in the literature, and have been implemented for different services (e.g., sponsored search, Google Ad Manager).

The procedure of a mechanism can be summarized briefly as follows: A principal first commits to a mechanism (i.e., rules of computing outcomes). Then all agents form a equilibrium where each of them computes her own strategy to maximize her utility in this mechanism based on her preference and knowledge about the environments (e.g., prior or belief about the nature or other agents). Finally, the outcome will be determined by the equilibrium.

Though the class of mechanisms is rich, the classical mechanism design literature mainly focus on a subclass of mechanisms, in which the strategy of the agents in the equilibrium is to reveal their preference truthfully. We name these mechanisms as the *revelation mechanisms*, and all other mechanisms as the *non-revelation mechanisms*. The aforementioned favor of revelation mechanism is due to a seminal observation – *revelation principle* as follows: if there is a mechanism with good equilibrium outcome, there is a revelation mechanism which achieves the same outcome in a truthtelling equilibrium. This constructed mechanism asks agents to reveal true preferences, simulates the agent strategies in the original mechanism, and outputs the outcome of the simulation.

In contrast to a vast number of studies in mechanism design focus on revelation mechanisms and the lack of theory of non-revelation mechanism due to the revelation principle, successful applications – e.g., first-price auction, generalized second-price auction – suggest a great practical impact for non-revelation mechanisms. As a recent example, Google recently move from a revelation mechanism (i.e., second-price auction) to a non-revelation mechanism (i.e., first-price auction) for its Google Ad Manager service (Google, 2019).

To address this discrepancy, in Section 1.1, we review a seminal argument – end-to-end principle – for the system design in computer science, and discuss its connection to the mechanism design. Next, we formalize such argument and work toward a new justification (i.e., *revelation gap*) for the potential inadequacy of revelation principle and the advantages of non-revelation mechanisms. See Section 1.2 and Section 1.3 for more discussions.

1.1. Multi-party Computation and End-to-end Principle

One important research direction in modern computer science focuses on multi-party computation. Two fundamental concerns in this area are (i) who should be doing what part of the computation; and (i) what are their incentives to do it correctly. The second concern has been studied extensively in the economics field of mechanism design. For the first concern, however, the system design field and the mechanism design field have different high-level guidelines.

From the view of multi-party computation, the mechanism itself as well as the participating agents can be thought as different parties in the system, where agents have their private preference as their own data. Revelation mechanisms correspond to systems where the computation is done by the central protocol (i.e. mechanism) and other parties (end points, i.e. agents) only truthfully reveal their data to the central protocol. Non-revelation mechanisms correspond to systems where agents are also perform some of the computation (i.e. computing their strategies). The system design literature advocates non-revelation mechanisms, as the *end-to-end principle* (cf. Saltzer et al., 1984) – a long-standing principle in this area – suggests that the computation should be done where the data is, and all environment-specific complexity should be push to the end points, i.e., in a decentralized fashion. This guiding principle was originally introduced in the design of computer network. In Saltzer et al. (1984), authors argue that following this principle, the computer network can be simple, robust (under various environments) and implemented under low cost. As a consequence, it enables the Internet protocols designed for the workloads of the 1980s to continue to succeed with workloads of the 2010s. On the other hand, due to the aforementioned revelation principle, the mechanism design literature favors revelation mechanisms. Generally speaking, since the revelation mechanisms are required to solve the complex task of finding an outcome that both enforces the truthfulness property and also obtains a desirable outcome, unsurprisingly, optimal revelation mechanisms tend to be complex and presumably fragile with dependent on the environment.

1.2. Prior-independent Mechanism Design and Revelation Gap

In this thesis, we work toward a formal argument that the decentralization idea from the end-to-end principle in the system design is beneficial even in purely economic terms when robust mechanisms are desired. In particular, we consider questions from *prior-independent mechanism design*, in which a mechanism is designed for agents with preferences drawn from an unknown distributions (a.k.a. prior). The goal is to identify robust mechanisms – ones with good (multiplicative) prior-independent approximation to the optimal mechanism that is tailored to the distribution of preferences. In prior-independent mechanism design, it is not generally without loss to restrict to revelation mechanisms – the equilibrium strategies for Bayesian agents in non-revelation mechanisms are a function of their prior and thus the construction of revelation mechanism via revelation principle is no longer prior-independent. Nonetheless, similar to other lines of research in mechanism design, most results in prior-independent mechanism design focus, with loss of generality, on revelation mechanisms.

To understand the loss of the restriction to revelation mechanisms, we introduce *revelation gap*, a quantification of optimal prior-independent approximation ratio among all revelation mechanisms vs. the optimal prior-independent approximation ratio among all (possibly non-truthful) mechanisms. A non-trivial revelation gap – that the prior-independent approximation factor of the best non-revelation mechanism is better than that of the best revelation mechanism – gives concrete motivation for a theory of mechanism design without the revelation principle.[1]

[1]It is not hard to invent pathological scenarios where there is a non-trivial revelation gap. Instead, this thesis considers the canonical environment of welfare maximization and revenue maximization.

1.3. Approach and Results

In this thesis, we study two canonical settings in prior-independent mechanism design and identify non-trivial revelation gaps for both welfare maximization and revenue maximization.

1.3.1. Welfare Maximization for Public Budgeted Agents

The first contribution of this thesis considers welfare maximization for agents with budgets and shows a non-trivial revelation gap for distributions on preferences that satisfy a standard regularity property. Moreover, the setting in which we exhibit the revelation gap suggests the end-to-end principle: the agents can easily implement the optimal outcome in the equilibrium of a simple mechanism, while revelation mechanisms that satisfy the constraints must be complex and either prior-dependent or non-optimal.

Our analysis focuses on welfare maximization in a canonical single-item environment with ex ante symmetric budget constrained agents, i.e., each agent's value is drawn independently and identically from an unknown distribution and the agent cannot make payments that exceed a known and identical budget (cf. Maskin, 2000). Our main treatment of this problem will make a simplifying assumption that the distribution follows a regularity property that implies that the Bayesian optimal mechanism has a nice form (Pai and Vohra, 2014). Our results require a symmetric environment, i.e., an i.i.d. distribution and identical budget.

The main challenge in demonstrating a revelation gap is that it is difficult to identify prior-independent optimal (revelation) mechanisms (cf. Fu et al., 2015) for non-trivial environments. This question remains as the open problem for a long period, until Allouah and

Besbes (2018); Hartline et al. (2020) showed that the second-price auction (resp. a variant of second-price auction, i.e., markup mechanism) is the prior-independent optimal revelation mechanism for revenue when the two agents' values are distributed according to a monotone hazard rate (resp. regular) distribution. Our non-trivial revelation-gap theorem follows from three results. First, the all-pay auction (from the literature, defined below) has a unique equilibrium that is Bayesian optimal and it is prior-independent. Second, we obtain a lower bound on the ability of a prior-independent revelation mechanism to approximate the Bayesian optimal mechanism by identifying the dominant strategy incentive compatible mechanism that is Bayesian optimal for the uniform distribution. The performance of this mechanism is strictly worse than that of the Bayesian optimal mechanism (which is Bayesian incentive compatible); specifically the gap is 1.013. [2] Third, we show that the dominant strategy incentive compatible clinching auction (from the literature, defined below) is an $e \approx 2.72$ approximation to the Bayesian optimal mechanism.

Combining the upper and lower bounds we see a revelation gap between 1.013 and e. The first result follows naturally from the literature; the second and third results are the main technical contributions of the paper. See Table 1.1 for a summary of all three results.

Three auctions are at the forefront of our study. The *all-pay auction* solicits bids, assigns the item to the highest agent, and charges all agents their bids. The *clinching auction* (Ausubel, 2004; Dobzinski et al., 2008; Goel et al., 2015) is an ascending price auction that can be thought of as allocating a unit measure of lottery tickets: a price is offered

[2] To better appreciate the magnitude of this lower bound, notice that it is demonstrated for two agents with uniformly distributed values where the optimal expected welfare (even without budgets) is is $2/3$ and the lottery mechanism (which gives the item to a random agent) has expected welfare $1/2$ and is a $4/3 \approx 1.33$ approximation.

Table 1.1. Prior-independent approximation ratio of welfare maximization for i.i.d. public budgeted agents. We consider the class of *public-budget regular* distributions (i.e., distributions with concave cumulative function).

	Class of revelation mechanisms	Class of all mechanisms
Upper bound	$e^{(*)}$	$1^{(\dagger)}$
Lower bound	$1.013^{(\ddagger)}$	$1^{(\dagger)}$

$^{(*)}$ Theorem 3.8; $^{(\dagger)}$ Maskin (2000); $^{(\ddagger)}$ Lemma 3.15 and Lemma 3.17.

in each stage, each agent specifies the measure of tickets desired at the given price, each agent is allocated a number of tickets that is equal to the minimum of her demand and the measure of remaining tickets if this agent is only allowed to buy tickets after all other agents have bought as much as they desire first. [3] The *middle-ironed clinching auction* – which we identify as the optimal dominant strategy incentive compatible mechanism – behaves like the clinching auction except that values that fall within a middle range are ironed. The allocation that an agent in this middle range receives is the average over he original allocation of for middle range values in the clinching auction. This averaging results in the budget binding later and more efficient outcomes than in the original clinching auction.

The second step, mentioned above, is to obtain a lower bound on the prior-independent approximation of a revelation mechanism. Our analysis begins with the observation that a prior-independent revelation mechanism must be Bayesian incentive compatible for every distribution. For two agents, this condition is equivalent to being dominant strategy

[3]For example, at a price of 0 all agents would want to buy all the tickets, but the agent that arrives last gets no tickets, thus no agents get any tickets at this price; the price increases.

incentive compatible. We ask whether there a gap between the Bayesian optimal dominant strategy and Bayesian incentive compatible mechanism. The comparison between optimal dominant strategy and Bayesian incentive compatible mechanism is standard for multi-dimensional mechanism design problems (e.g., see Gershkov et al., 2013; Yao, 2017); we are unaware of previous studies of this phenomenon for single-dimensional agents with non-linear preferences. We answer this question positively by writing the dominant strategy mechanism design problem as a linear program and solving it by identifying a dual solution that proves the optimality of the middle-ironed clinching auction (cf. Pai and Vohra, 2014; Devanur and Weinberg, 2017). The identified gap gives a lower bound on the approximation factor of the optimal prior-independent mechanism.

The third step, mentioned above, proves that the prior-independent approximation factor of the clinching auction auction is at most e and resolves in the affirmative an open question from Devanur et al. (2013). Our proof follows from a novel adaptation of a standard method for approximation results in mechanism design where an auction's performance is compared to the upper bound given by the ex ante relaxation, in this case, the welfare of the optimal mechanism that sells one item in expectation over the random draws of the agents' values (i.e., ex ante) rather than for all draws of the agents' values (i.e., ex post). This method was introduced by Chawla et al. (2007), formalized by Alaei (2011, 2014), generalized by Alaei et al. (2013), and employed in many subsequent analyses.

1.3.2. Revenue Maximization for A Linear Agent with A Single Sample Access

The second contribution of this thesis considers revenue maximization in a canonical single-item environment for a single agent with a single sample access, i.e., the agent's value is drawn from an unknown distribution but the mechanism can access a single sample (independent to agent's value) from that distribution (cf. Dhangwatnotai, Roughgarden, and Yan, 2015; Allouah and Besbes, 2019). The agent knows her private valuation and the distribution for valuation, but she does not know the sample of the mechanism. Our main theorem identifies a non-trivial revelation gap for revenue maximization in this model.

This revelation gap for revenue maximization follows from three results. First, we introduce the (non-revelation) sample-bid mechanism and obtain an upper bound of its prior-independent approximation ratio. Second, we obtain a lower bound of the optimal prior-independent approximation ratio among all possible mechanisms. Third, we show that any revelation mechanism[4] is equivalent to a sampled-based pricing mechanism introduced by Allouah and Besbes (2019) where the authors lower-bound and upper-bound the optimal prior-independent approximation ratio among all sample-based pricing mechanisms. See Table 1.2 for a summary of all three results. Since the prior-independent approximation ratio of the sample-bid mechanism is strictly better than the optimal prior-independent approximation ratio among all revelation mechanisms, we immediately get our non-trivial revelation gap for revenue maximization.

[4]We impose a technical assumption (i.e. scale-invariant) to the class of revelation mechanisms, which is common in prior-independent mechanism design (Allouah and Besbes, 2018, 2019; Hartline, Johnsen, and Li, 2020).

Table 1.2. Prior-independent approximation ratio of revenue maximization for a linear agent with a single sample access. Two class of distributions (i.e. *regular*, *MHR* – standard assumptions in mechanism design) are considered, where MHR distributions is a subclass of regular distributions. We impose a technical assumption (i.e. scale-invariant, Definition 4.11) to the class of revelation mechanisms.

	Class of revelation mechanisms		Class of all mechanisms	
	Regular dists.	MHR dists.	Regular dists.	MHR dists.
Upper bound	$1.996^{(*)}$	$1.575^{(*)}$	$1.835^{(\S)}$	$1.296^{(\dagger)}$
Lower bound	$1.957^{(*)}$	$1.543^{(*)}$	$1.073^{(\ddagger)}$	

$^{(*)}$ Allouah and Besbes (2019) and Lemma 4.28; $^{(\S)}$ Theorem 4.7; $^{(\dagger)}$ Theorem 4.11; $^{(\ddagger)}$ Theorem 4.23.

In the model of a single agent with single-sample access, the class of non-revelation mechanisms is rich, which includes fairly complicated mechanisms. For example, mechanisms can ask agents to reports both her value and prior; or include multiple rounds of communication between seller and agent who sequentially reveal their private information.[5] Nonetheless, our upper bound of the optimal prior-independent approximation ratio is attained by a simple non-revelation mechanism – *sample-bid mechanism* defined as follow.

- *Sample-bid mechanism:* Given parameter α and sample s, the sample-bid mechanism solicits a non-negative bid $b \geq 0$, charges the agent $\alpha \cdot \min\{b, s\}$, and allocates the item to the agent if $b \geq s$.

From the agent's perspective, she reports a bid to compete for the item against a random sample realized from the same valuation distribution; and regardless of whether she wins

[5]Recall that the agent knows the distribution of the sample but does not know its realization.

or loses, she will always be charged $\alpha \cdot \min\{b, s\}$. In fact, the agent's optimal bidding strategy could be overbidding or underbidding, depending on the value as well as the distribution. The sample-bid mechanism has the similar format as the Becker–DeGroot–Marschak method (Becker, DeGroot, and Marschak, 1964) which has been studied and implemented in experimental economics for understanding agents' perception of the random event.

In order to beat the optimal prior-independent approximation ratio among all revelation mechanisms, we need to show the approximation for the sample-bid mechanism is strictly better than $1.957 < 2$ for regular distributions, and $1.543 < {}^{e}/_{(e-1)}$ for MHR distributions. However, most approximation techniques and results for non-revelation mechanisms in the literature only provide similar or or larger constants – for instance, smoothness property, permeability, and revenue covering property in price of anarchy (cf. Roughgarden, Syrgkanis, and Tardos, 2017; Dütting and Kesselheim, 2015; Hartline, 2016, see more discussion in related work).[6] One the other hand, analyzing the approximation of revelation mechanisms is relatively easier. In revenue maximization, one analysis approach used extensively for revelation mechanisms is the revenue curve reduction (see next paragraph). This approach has lead to tight or nearly tight results in both prior-independent approximation (Allouah and Besbes, 2018, 2019; Hartline, Johnsen, and Li, 2020) and Bayesian approximation (Alaei, Hartline, Niazadeh, Pountourakis, and Yuan, 2018; Jin, Lu, Tang, and Xiao, 2019b; Jin, Lu, Qi, Tang, and Xiao, 2019a).

Revenue curves (cf. Bulow and Roberts, 1989) give an equivalent representation of agent's valuation distribution and enable clean characterizations of the revenue of any mechanism (see e.g. Myerson, 1981; Bulow and Roberts, 1989; Alaei, Fu, Haghpanah, and

[6] Feng and Hartline (2018) bypass this challenge in their revelation gap for welfare maximization by considering a model where the all-pay auction (cf. Maskin, 2000) achieves prior-independent approximation ratio 1, i.e., it is indeed the Bayesian optimal mechanism.

Hartline, 2013). The high-level goal of revenue curve reduction is to identify a subclass of revenue curves that has closed form and over which the worst approximation guarantee is attained. The main argument is to design a (problem or mechanism) specific modification to the revenue curve (converting an arbitrary revenue curve into a revenue curve from the subclass) and analyze the impact of revenue from the modification on the given mechanism. Note that revenue is the expected payment of the agents when they bid optimally. For revelation mechanisms, after the modification has been designed, it is sufficient to study how payment changes for every bid in the modification, since agents are bidding revelationly (i.e. bids equal values). However, for non-revelation mechanisms, converting a revenue curve to another one will lead to changes in both the payment for each bid and the optimal bidding strategy of each agent. This makes the revenue curve reduction approach more difficult for non-revelation mechanisms, and thus, results of non-revelation mechanisms in the literature rarely uses this technique. In this paper, due to the simplicity of our model and the sample-bid mechanism, we are able to apply this technique by carefully (but relatively loosely) disentangling these two impacts and then analyzing them separately.

Our final result for the single-agent pricing from samples model provides a lower bound on the optimal prior-independent approximation ratio among the class of all mechanisms. This result contrasts with multi-agent models where there there exists complicated and arguably impractical non-revelation mechanism whose prior-independent approximation is arbitrarily close to 1.[7] The crucial observation for proving this lower bound is that for pointmass distributions, the agent perfectly knows the seller's sample. Thus, she can

[7]Such mechanisms are designed and analyzed in non-parametric implementation theory – a line of research in economics, see the survey of Jackson (2001) and further discussion in the related work section.

strategically imitate the behavior of the values in other distributions. This restricts the seller's ability to extract revenue from the agent, which leads to a prior-independent approximation ratio at least 1.073 even on the restricted subclass of MHR distributions (in fact, even on uniform distributions). Our lower bound also suggests that it will be non-trivial to identify the non-revelation mechanism which attains the optimal prior-independent approximation ratio.

It should be noted that our better-performing prior-independent non-revelation mechanisms do not come without drawbacks relative to prior-independent revelation mechanisms. Elegantly, prior-independent revelation mechanisms do not require prior knowledge by any party. In contrast, prior-independent non-revelation mechanisms generally require some knowledge of the prior on the part of the agents. From this perspective, our results show that a seller is able to extract strictly higher revenue from the agent by taking advantage of information that the agent possesses and is able to strategize with respect to. This echoes the argument in the end-to-end principle that decentralized system can achieve higher performance since the end points (i.e., agents) have more information than the central protocol (i.e., mechanisms).

1.4. Related Work

Prior-independent Mechanism Design. As a standard framework for understanding the robustness of mechanisms, prior-independent mechanism design has been applied to single-dimensional mechanism design (Dhangwatnotai et al., 2015; Roughgarden et al., 2012; Fu et al., 2015; Allouah and Besbes, 2018; Hartline et al., 2020), multi-dimensional mechanism design (Devanur et al., 2011; Roughgarden et al., 2015; Goldner and Karlin, 2016), makespan minimization (Chawla et al., 2013), mechanism design for risk-averse agents (Fu et al., 2013), and mechanism design for agents with interdependent values (Chawla et al., 2014a). Except Fu et al. (2013), all other results focus on truthful mechanisms.

Non-revelation Mechanism Design. Some recent works non-revelation mechanism design are equilibrium analysis of i.i.d. rank-based mechanism (Chawla and Hartline, 2013), robust analysis of welfare and revenue for classic mechanisms in practice (i.e. price of anarchy, see next paragraph), estimating revenue and welfare in a mechanism from equilibrium bids in another mechanism (Chawla et al., 2014b, 2016), and the sample complexity of non-truthful mechanisms in asymmetric environments (Hartline and Taggart, 2019, mentioned above).

Price of anarchy studies how classic non-truthful mechanisms (e.g. first-price auction, all-pay auction) approximate the optimal welfare. Syrgkanis and Tardos (2013) introduce a smoothness property defined on mechanisms and give an analysis framework based on this property. With this smoothness framework, the authors upper-bound the welfare-approximation of the first-price auction by ${}^{e}/_{(e-1)}$, and the welfare-approximation of the all-pay auction by 2. These two results are later tightened by Christodoulou et al. (2015) for the all-pay auction and Hoy et al. (2018) for the first-price auction using some mechanism-specific arguments. Hartline et al. (2014) introduce a geometric framework for analyzing the price of anarchy for both welfare and revenue. As the instantiations of the framework, authors upper-bound the revenue approximation of the first-price auction with individual monopoly reserve by ${}^{2e}/_{(e-1)}$. Dütting and Kesselheim (2015) show that bounds from these analysis frameworks are tight up to constant factors.

Non-parametric Implementation Theory. The literature on non-parametric implementation theory considers the same question as prior-independent mechanism design but allows mechanisms where agents cross-report their beliefs on other agents' values (e.g., Jackson, 2001). Caillaud and Robert (2005) introduce a dynamic auction for single-item multi-agent settings which is able to implement the Bayesian revenue optimal auction (Myerson, 1981) without the knowledge of agents' distribution. Dasgupta and Maskin (2000) introduce a generalization of VCG auction for multi-agent interdependent value settings.[8] In this auction, agents are asked to submit a function that gives a bid for every possible valuation of the other agents. Though this auction requires no knowledge of agents' distributions, Dasgupta and Maskin (2000) show that it is Bayesian welfare-optimal under mild assumptions. Azar et al. (2012) study how to use scoring rules to learn agents' distribution and implement the auction based on this learned distribution. All results above suggest that in the multi-agent settings, there exist complicated and arguably impractical non-truthful mechanisms whose prior-independent approximation equal or are arbitrarily close to 1. However, as we mentioned earlier, in the model of a single-agent with single-sample access, we provide a lower bound on the optimal prior-independent approximation without any restriction on mechanisms.

Mechanism Design for Agents with Budget Constraints. Mechanism design for agents with budget constraint is well studied in the literature. Laffont and Robert (1996) and Maskin (2000) study the revenue-maximization and welfare-maximization problems for symmetric agents with *public* budgets in single-item environments. Boulatov and

[8] In general, there is no incentive compatible mechanism which outputs the welfare-optimal outcomes in interdependent value settings.

Severinov (2018) generalize their results to agents with i.i.d. values but asymmetric public budgets.

Che and Gale (2000) consider the single agent problem with *private* budget and valuation distribution that satisfies declining marginal revenues, and characterize the optimal mechanism by a differential equation. Devanur and Weinberg (2017) consider the single agent problem with private budget and an arbitrary valuation distribution, characterize the optimal mechanism by a linear program, and use an algorithmic approach to construct the solution. Pai and Vohra (2014) generalize the characterization of the optimal mechanism to symmetric agents with uniformly distributed private budgets. Richter (2016) shows that a price-posting mechanism is optimal for selling a divisible good to a continuum of agents with private budgets if their valuations are regular with decreasing density. For more general settings, no closed-form characterizations of optimal mechanism are known. However, the optimal mechanism can be solved by a polynomial-time solvable linear program over interim allocation rules (cf. Alaei et al., 2012; Che et al., 2013). Feng et al. (2020) introduce a framework to extend the approximation of any deterministic and dominant strategy incentive compatible mechanism for agents with linear utility to agents with non-linear utility in single-item environments for revenue maximization. As an instantiations of the framework, they give small constant bounds on the simple mechanisms introduced in the literature for agents with budget constraints under various assumptions.

Mechanism Design with Sample Access. There is a significant area of research studying mechanism design with sample access from the distribution of agents' preference, which has two regimes – small number of samples, and large number of samples. In the former regime, literature studies the approximation of mechanisms with a single-sample access (Azar et al., 2014; Dhangwatnotai et al., 2015; Allouah and Besbes, 2019; Feng et al., 2019; Correa et al., 2019; Dütting et al., 2020; Correa et al., 2020), and mechanisms with two-sample access (Babaioff et al., 2018; Daskalakis and Zampetakis, 2020). In the latter regime, the goal is to minimize the sample complexity, i.e., number of sample to achieve $(1-\epsilon)$-approximation (e.g. Cole and Roughgarden, 2014; Morgenstern and Roughgarden, 2015; Huang et al., 2018; Gonczarowski and Weinberg, 2018; Guo et al., 2019; Hartline and Taggart, 2019). Except Hartline and Taggart (2019), all other results focus on truthful mechanisms.

1.5. Organization of the Thesis

In Chapter 2, we introduce the classic single-item auction problem in Bayesian mechanism design, and its extension to prior-independent mechanism design; and discuss the preliminary results, technique, and the formal definition of the revelation gap. In Chapter 3, we consider single-item auction for public budgeted agents, and show a non-trivial revelation gap for welfare-maximization in this setting. In Chapter 4, we shift our attention to single-item auction for a linear agent with a single sample access, and show non-trivial revelation gaps for revenue-maximization.

1.6. Bibliographic Notes

The content in this thesis is based on two research papers with co-authors: "*An end-to-end argument in mechanism design (prior-independent auctions for budgeted agents)*" by Feng and Hartline (2018) and "*Revelation Gap for Pricing from Samples*" by Feng, Hartline, and Li (2021).

CHAPTER 2

Model and Preliminaries

2.1. Bayesian Mechanism Design

In this thesis, we consider the classic single-item auction problem where a seller wants to allocate an item to n agents.

Agent Models. Each agent $i \in [n]$ has a private value $v_i \in \mathbb{R}_{\geq 0}$ drawn from the valuation distribution (a.k.a. prior) F supported on $[\underline{v}, \overline{v}]$. and a budget $w \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.[1] We assume that distribution F has positive density f every where in the support. Both the valuation distribution F and budget w are known by seller and agents. Given an allocation $x \in [0, 1]$ and payment $p \in \mathbb{R}_{\geq 0}$, the utility $u_w(v, x, p)$ of an agent with value v and budget w is [2]

$$u_w(v, x, p) = \begin{cases} vx - p & \text{if } p \leq w \ ; \\ -\infty & \text{otherwise .} \end{cases}$$

We say an agent has *linear utility* if $w = \infty$; and has *public budgeted utility* if w is finite. We focus on single-item environments where the seller can allocate at most one unit of the item to all agents, i.e., an allocation profile $\boldsymbol{x} = (x_1, \ldots, x_n)$ is feasible if and only if $\sum_{i \in [n]} x_i \leq 1$. We denote the set of all feasible allocation profiles as $\mathcal{X}$.

Mechanisms. A *(sealed-bid) mechanism* $\mathcal{M} = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ is given by mappings from bid profile $\boldsymbol{b} = (b_1, \ldots, b_n)$ to allocations and payments, which we denote by $\tilde{\boldsymbol{x}}(\boldsymbol{b}) = (\tilde{x}_1(\boldsymbol{b}), \ldots, \tilde{x}_n(\boldsymbol{b}))$

[1]We focus on symmetric agents, i.e., all agents have the same valuation distribution F and budget w.
[2]We set budget w as the subscript of utility function u, since it is encoded in the utility function, instead of the type. We often omit the subscript w in the utility function if it is clear from the context.

and $\tilde{\boldsymbol{p}}(\boldsymbol{b}) = (\tilde{p}_1(\boldsymbol{b}), \dots, \tilde{p}_n(\boldsymbol{b}))$. We assume that for every bid profiles $\boldsymbol{b}$, allocations are feasible, i.e., $\boldsymbol{x}(\boldsymbol{b}) \in \mathcal{X}$; and allocations, payments are non-negative, i.e., $\tilde{x}_i(\boldsymbol{b}) \geq 0, \tilde{p}_i(\boldsymbol{b}) \geq 0$ for every agent $i \in [n]$.

Example 2.1. *Two classic mechanisms are studied and implemented in both theory and practice – the* second-price auction *and the* all-pay auction*. In both auctions, the allocation rule is* highest-bids-win*, i.e.,* $\tilde{\boldsymbol{x}}(\boldsymbol{b}) \in \operatorname{argmax}_{\boldsymbol{x} \in \mathcal{X}} \sum_i b_i x_i$*, and the payment rule is defined as follows:*

- *in the second-price auction,* $\tilde{p}_i(\boldsymbol{b}) = \min\{b : \tilde{x}_i(b, \boldsymbol{b}_{-i}) = 1\} \cdot \tilde{x}_i(\boldsymbol{b})$*;*
- *in the all-pay auction,* $\tilde{p}_i(\boldsymbol{b}) = b_i$*.*

Equilibrium. A strategy σ_i for agent i is a mapping from her value v_i to her reported bid b_i. A strategy profile $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ forms a *Bayes-Nash equilibrium (BNE)* in a mechanism $\mathcal{M}$ if each agent i's strategy σ_i maximizes her expected utility in $\mathcal{M}$ given that all other agents play their own strategies in the strategy profile $\boldsymbol{\sigma}$, i.e., for every agent i, value v_i and bid b_i,

$$\mathbf{E}_{\boldsymbol{v}_{-i}}[u(v_i, x_i(\boldsymbol{\sigma}(\boldsymbol{v})), p_i(\boldsymbol{\sigma}(\boldsymbol{v})))] \geq \mathbf{E}_{\boldsymbol{v}_{-i}}[u(v_i, x_i(b_i, \boldsymbol{\sigma}_{-i}(\boldsymbol{v}_{-i})), x_i(b_i, \boldsymbol{\sigma}_{-i}(\boldsymbol{v}_{-i})))]$$

where $\boldsymbol{\sigma}(\boldsymbol{v}) = (\sigma_1(v_1), \dots, \sigma_n(v_n))$. Similarly, a strategy profile $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ forms a *dominant strategy equilibrium (DSE)* in a mechanism $\mathcal{M}$ if each agent i's strategy σ_i maximizes her ex post utility in $\mathcal{M}$ regardless of other agents' bids, i.e., for every agent i, value v_i, bid b_i, and other agents bids $\boldsymbol{b}_{-i}$

$$u(v_i, x_i(\sigma_i(v_i), \boldsymbol{b}_{-i}), p_i(\sigma_i(v_i), \boldsymbol{b}_{-i})) \geq u(v_i, x_i(b_i, \boldsymbol{b}_{-i}), x_i(b_i, \boldsymbol{b}_{-i}))$$

Note that by definition every DSE is also a BNE.

When a mechanism $\mathcal{M} = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ induces a unique BNE $\boldsymbol{\sigma}$, we define the *ex post allocation rule* and *ex post payment rule* (in value space) as $x_i(\boldsymbol{v}) = \tilde{x}_i(\boldsymbol{\sigma}(\boldsymbol{v}))$ and $p_i(\boldsymbol{v}) = \tilde{p}_i(\boldsymbol{\sigma}(\boldsymbol{v}))$; *interim allocation rule* and *interim payment rule* as $x_i(v_i) = \mathbf{E}_{v_{-i}}[x_i(\boldsymbol{v})]$ and $p_i(v_i) = \mathbf{E}_{v_{-i}}[p_i(\boldsymbol{v})]$.

Example 2.2. *Consider agents with linear utilities.*

- *In the second-price auction, revealing agents' own values truthfully (i.e., $\sigma_i(v_i) = v_i$) is the unique DSE and BNE. Hence, the item is always allocated to the agent with the highest value, which implies that the interim allocation rule is $x_i(v_i) = (F(v_i))^{n-1}$, and the interim payment rule is $p_i(v_i) = \int_{\underline{v}}^{v_i} t \cdot f(t)(F(t))^{n-2}\, dt$.*
- *In the all-pay auction, the unique BNE is $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_i(v_i) = \int_{\underline{v}}^{v_i} t \cdot f(t)(F(t))^{n-2}\, dt$. Hence, the item is always allocated to the agent with the highest value, which implies that the interim allocation rule is $x_i(v_i) = (F(v_i))^{n-1}$, and the interim payment rule is $p_i(v_i) = \int_{\underline{v}}^{v_i} t \cdot f(t)(F(t))^{n-2}\, dt$.*

In Example 2.2, it is not a coincidence that the second-price auction and the all-pay auction have the same interim payment rule, instead it is a consequence of the characterization of BNE from Myerson (1981).

Theorem 2.1 (Myerson, 1981)**.** *A strategy profile $\boldsymbol{\sigma}$ are in BNE (resp. DSE) if and only if for all agent i,*

- (monotonicity) *$x_i(v_i)$ (resp. $x_i(v_i, \boldsymbol{v}_{-i})$) is monotone non-decreasing in v_i (resp. for all $\boldsymbol{v}_{-i}$);*

- (payment identify) $p_i(v_i) = v_i x_i(v_i) - \int_{\underline{v}}^{v_i} x_i(t)\,dt + p_i(\underline{v})$ *(resp.* $p_i(v_i, \boldsymbol{v}_{-i}) = v_i x_i(v_i, \boldsymbol{v}_{-i}) - \int_{\underline{v}}^{v_i} x_i(t, \boldsymbol{v}_{-i})\,dt + p_i(\underline{v}, \boldsymbol{v}_{-i})$*).*

Incentive Compatibility. A mechanism $\mathcal{M}$ is *Bayes incentive compatible (BIC)* if revealing agents' own values truthfully (i.e., $\sigma_i(v_i) = v_i$) is a BNE. Similarly, a mechanism $\mathcal{M}$ is *dominant strategy incentive compatible (DSIC)* if revealing agents' own values truthfully (i.e., $\sigma_i(v_i) = v_i$) is a DSE.[3] We also name BIC (and thus DSIC) mechanisms as *revelation mechanisms*; and other mechanisms as *non-revelation mechanisms*.

Remark 2.3. *As we illustrate in Example 2.2, for agents with linear utilities, the second-price auction is a revelation mechanism, while the all-pay auction is a non-revelation mechanism.*

Bayesian Optimal Mechanisms. In this paper, we study the objective of *welfare* and *revenue*. The welfare of a mechanism is $\mathbf{E}_{\boldsymbol{v}}[\sum_i v_i x_i(\boldsymbol{v})]$. The revenue of a mechanism is $\mathbf{E}_{\boldsymbol{v}}[\sum_i p_i(\boldsymbol{v})]$.

To analysis the revenue, Myerson (1981) introduces the *virtual value* characterization, and shows the equivalence between virtual surplus and revenue.

Definition 2.4. *The* virtual value *of an agent with value v drawn from valuation distribution F is $\phi(v) = v - \frac{1-F(v)}{f(v)}$.*

Lemma 2.2 (Myerson, 1981)**.** *For any interim allocation rule $x(\cdot)$ and interim payment rule $p(\cdot)$, the expected revenue equals the expected virtual surplus, i.e.,* $\mathbf{E}_v[\phi(v)x(v)] + p(\underline{v}) = \mathbf{E}_v[p(v)]$.

[3]Note that by definition, any DSIC mechanism is BIC.

Fixing a valuation distribution F, a mechanism is *Bayesian revenue-optimal* (resp. *Bayesian welfare-optimal*) if its expected revenue (resp. welfare) in the equilibrium with respect to valuation distribution F is optimal. When the context is clear, we use *Bayesian optimal* for short.

Remark 2.5. *As we illustrate in Example 2.2, for agents with linear utilities, both the second-price auction and the all-pay auction are Bayesian welfare-optimal.*

Example 2.6. *The second-price auction with reserve* $\phi^{-1}(0)$ *is a DSIC mechanism with allocation rule* $\tilde{\boldsymbol{x}}(\boldsymbol{b}) \in \mathrm{argmax}_{\boldsymbol{x}\in\mathcal{X}} \sum_i \phi(v_i)x_i$*. For agents with linear utilities and* regular *valuation distribution* F *(i.e.,* $\phi(v) = v - \frac{1-F(v)}{f(v)}$ *is non-decreasing in* v*), the second-price auction with reserve* $\phi^{-1}(0)$ *is the Bayesian revenue-optimal.*

Revelation Principle. The following observation – *revelation principle* from Myerson (1981) ensures that it is without loss to assume that the Bayesian optimal mechanism is BIC.[4]

Proposition 2.3 (revelation principle, Myerson, 1981)**.** *If there exists an arbitrary mechanism that induces a equilibrium with interim allocation* $\boldsymbol{x}$ *and interim payment* $\boldsymbol{p}$*; then there there exists another revelation mechanism that induces a truthtelling equilibrium with the same interim allocation* $\boldsymbol{x}$ *and interim payment* $\boldsymbol{p}$*.*

The proof of the revelation principle is simple: A revelation mechanism can simulate the equilibrium strategies in the non-revelation mechanism to obtain the same outcome as

[4]For agents with linear utilities, it is without loss to assume that the Bayesian optimal mechanism is DSIC. Manelli and Vincent (2010) show that fixing any valuation distribution F and any BIC mechanism, there is a DSIC mechanism that achieves the same interim allocation and payment as in the BIC mechanism. For agents with public budgeted utility, as we shown in Chapter 3, the Bayesian optimal mechanism may not be implemented as a DSIC mechanism.

a truthtelling equilibrium, i.e., agents reveal true values to the revelation mechanism, it simulates the agent strategies in the non-revelation mechanism, and it outputs the outcome of the simulation. In particular, for Bayesian mechanism design (where the agents' values are drawn from F), the agents' equilibrium strategies are a function of F and thus the corresponding revelation mechanism constructed via the revelation principle relays on the knowledge of F.

2.2. Prior-independent Mechanism Design

One crucial assumption in Bayesian mechanism design is the mechanism designer's knowledge of valuation distribution (a.k.a. prior) F. For instance, the Bayesian revenue-optimal mechanism in Example 2.6 is a prior-dependent mechanism. To measure how the performance degrades due to the lack of the knowledge of prior, literature introduces prior-independent mechanism design.

Definition 2.7. *Fixing an objective (e.g., welfare, revenue), a class of distributions DISTS, the* prior-independent approximation ratio $\Gamma(\mathcal{M}, \mathit{DISTS})$ *of a mechanism* $\mathcal{M}$ *is defined as*

$$\Gamma(\mathcal{M}, \mathit{DISTS}) = \max_{F \in \mathit{DISTS}} \frac{\mathbf{E}_{\boldsymbol{v} \in F^n}[\mathrm{OPT}_F(\boldsymbol{v})]}{\mathbf{E}_{\boldsymbol{v} \in F^n}[\mathcal{M}(\boldsymbol{v})]}.$$

where $\mathbf{E}_{\boldsymbol{v} \in F^n}[\mathcal{M}(\boldsymbol{v})]$ *is the objective value of mechanism* $\mathcal{M}$ *in equilibrium when agents values are drawn from valuation distribution* F*; and* $\mathbf{E}_{\boldsymbol{v} \in F^n}[\mathrm{OPT}_F(\boldsymbol{v})]\, F$ *is the objective value in the Bayesian optimal mechanism for distribution* F*.*

Note that the prior-independent approximation factor $\Gamma(\mathcal{M}, \mathrm{DISTS})$ is at least 1 by definition.

A mechanism $\mathcal{M}^*$ is *prior-independent optimal* within a class of mechanisms MECHS for a class of distributions DISTS, if its prior-independent approximation ratio is minimal, i.e.,

$$\mathcal{M}^* = \operatorname{argmin}_{\mathcal{M} \in \text{MECHS}} \mathit{\Gamma}(\mathcal{M}, \text{DISTS})$$

We denote $\min_{\mathcal{M} \in \text{MECHS}} \mathit{\Gamma}(\mathcal{M}, \text{DISTS})$ by $\mathit{\Gamma}(\text{MECHS}, \text{DISTS})$. When the class of distributions DISTS is clear from the context, we simplify $\mathit{\Gamma}(\text{MECHS}, \text{DISTS})$ as $\mathit{\Gamma}(\text{MECHS})$.

2.3. Revelation Gap

Recall that in the revelation principle for Bayesian mechanism design, the constructed revelation mechanism relays on the knowledge of prior. Thus, the revelation principle is no longer holds in prior-independent mechanism design. Thus, to quantify the optimal prior-independent approximation ratio among all revelation mechanisms vs. the optimal prior-independent approximation ratio among all (possibly non-revelation) mechanisms, we define the *revelation gap* as follows.

Definition 2.8 (revelation gap)**.** *Fixing an objective and a class of distribution* DISTS*, the* revelation gap $\rho(\text{DISTS})$ *is the ratio between the optimal prior-independent approximation factor within all revelation (i.e., BIC) mechanisms* $\mathbb{M}_r$ [5] *and the optimal prior-independent approximation factor within all mechanisms* $\mathbb{M}$*, i.e.,*

$$\rho(\text{DISTS}) \triangleq \frac{\mathit{\Gamma}(\mathbb{M}_r, \text{DISTS})}{\mathit{\Gamma}(\mathbb{M}, \text{DISTS})}.$$

[5]Recall that Bayesian incentive compatibility is defined with respect to a valuation distribution. Thus, the class of revelation mechanisms $\mathbb{M}_r$ considers all mechanisms that are BIC for all distributions. By definition, $\mathbb{M}_r$ contains all DSIC mechanisms.

Note that the revelation gap ρ(DISTS) is at least 1 by definition.

Example 2.9. *Consider welfare maximization for agents with linear utilities. Note that the second-price auction is a prior-independent, DSIC and Bayesian welfare-optimal. Thus, for any class of distributions DISTS, the revelation gap is trivial, i.e.,* $\rho(DISTS) = 1$.

In Chapter 3 and Chapter 4, we study two canonical settings and show non-trivial revelation gap for both welfare maximization and revenue maximization.

CHAPTER 3

Welfare Maximization for Public Budgeted Agents

In this chapter, we focus on prior-independent mechanism design for welfare maximization with public budgeted agents. In particular, we consider the class of distributions that satisfies the *public-budget regularity* defined as follows.

Definition 3.1. *A valuation distribution F is* public-budget regular *if its cumulative function is concave. An public-budgeted agent with such valuation distribution F is public-budget regular.*

In Section 3.1, we focus on the all-pay auction, which is a non-revelation mechanism and attains the optimal prior-independent approximation ratio, i.e., $\Gamma(\mathbb{M})$. In Sections 3.2 and 3.3, we focus on the clinching auction and its variant (i.e., the middle-ironed clinching auction), which are both revelation mechanisms, and provide upper and lower bounds on the optimal prior-independent approximation ratio for revelation mechanisms, i.e., $\Gamma(\mathbb{M}_r)$. Combining all results from Sections 3.1 to 3.3, we establish the non-trivial revelation gap in welfare maximization for public budgeted agents in Section 3.4. Finally, in Section 3.5, we discuss the prior-independent approximation ratios of the all-pay auction and the clinching auction for the class of all distributions (i.e., without public-budgeted regularity assumption).

3.1. Welfare of the All-pay Auction

For welfare maximization, Maskin (2000) characterizes the Bayesian optimal mechanisms for agents with public budget utilities. In particular, with the public-budget regularity assumption, the Bayesian optimal mechanism has a nice form.

The results of Maskin can be reinterpreted, à la Alaei et al. (2013), as solving a single-agent interim optimization problem that is given by an interim constraint $x^{\star}(\cdot)$. An interim allocation is interim feasible under the interim constraint $x^{\star}(\cdot)$ if for all values $v \in [\underline{v}, \overline{v}]$, the probability of allocating item to an agent with value greater than v with allocation rule $x(\cdot)$ is at most that with allocation rule $x^{\star}$, i.e.$\int_{v}^{\overline{v}} x(t)\, dF(t) \leq \int_{v}^{\overline{v}} x^{\star}(t)\, dF(t)$. In many cases solution to these interim optimization problems will take the form of the original constraint with ironed interval and reserve. Ironing on arbitrary interval $[v^{\dagger}, v^{\ddagger}]$ corresponds to the distribution weighted averaging as follow, $x(v) = \int_{v^{\dagger}}^{v^{\ddagger}} x^{\star}(t)\, dF(t)$ for all $v \in [v^{\dagger}, v^{\ddagger}]$. Reserve at value $v^{\dagger}$ corresponds to rejecting all value below $v^{\dagger}$ as follows, $x(v) = 0$ for all $v \in [0, v^{\dagger}]$. As we already illustrated in Example 2.1, an important allocation constraint is that given by the *highest-bid-wins* allocation rule. The highest-bid-wins allocation rule for n agents and with values from cumulative distribution function F is $x^{\star}(v) = (F(v))^{n-1}$, e.g., for two agents with uniform values it is $x^{\star}(v) = v$.

Theorem 3.1 (Maskin, 2000; Alaei et al., 2013)**.** *For public-budget regular i.i.d.agents and interim allocation constraint $x^{\star}(\cdot)$, the welfare-optimal single-agent mechanism allocates as by $x^{\star}(\cdot)$ except that values in $[v^{\dagger}, \overline{v}]$ are ironed for some $v^{\dagger}$; and payments are given deterministically by the payment identity.*

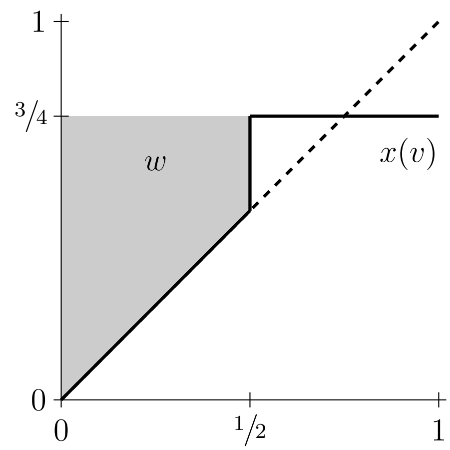


Figure 3.1. Depicted are the interim allocation rules of the welfare-optimal mechanism for two agents with uniform values on $[0, 1]$ and budget $w = 1/4$. In each figure the highest-bid-wins allocation rule is depicted with a dashed line.

Example 3.2. *For two agents with private value drawn from uniform distribution* $U[0, 1]$ *and budget* $w = 1/4$*, the welfare-optimal mechanism has allocation rule and payment rule as follows,*

$$x(v) = \begin{cases} v & \text{if } v \leq \frac{1}{2}, \\ \frac{3}{4} & \text{otherwise;} \end{cases} \qquad p(v) = \begin{cases} \frac{v^2}{2} & \text{if } v \leq \frac{1}{2}, \\ w & \text{otherwise.} \end{cases}$$

See Figure 3.1 for a graphical illustration.

For single-item environments, one possible implementation of Theorem 3.1 is the all-pay auction. The all-pay auction has a unique Bayes-Nash equilibrium which is identical to outcome described in Theorem 3.1 for the allocation constraint given by the highest-bid-wins allocation rule.

Definition 3.3 (all-pay auction)**.** *The* all-pay auction *is a mechanism* $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ *where* $\tilde{\boldsymbol{x}}(\cdot)$ *allocates item to the agent with highest bid with tie broken at random and* $\tilde{\boldsymbol{p}}(\cdot)$ *charges each agent their bid, i.e.,* $\tilde{p}_i(\boldsymbol{b}) = b_i$*.*

Theorem 3.2 (Maskin, 2000)**.** *For public-budget regular i.i.d.agents, the all-pay auction is welfare optimal.*

Note that the all-pay auction is prior-independent, and thus implies the following corollary on the optimal prior-independent approximation ratio.

Corollary 3.3. *For welfare maximization with public-budget regular i.i.d.agents, the optimal prior-independent approximation ratio* $\Gamma(\mathbb{M}) = 1$*, which is attained by the all-pay auction.*

3.2. Welfare of the Clinching Auction

In this section, we study a prior-independent revelation mechanism called the clinching auction in single-item environments. Dobzinski et al. (2008) gave the following formulation of the clinching auction and characterized properties of its outcome. See Figure 3.3b for a graphical illustration when there are two agents.

Definition 3.4 (clinching auction)**.** *The* clinching auction *maintains an allocation and price-clock starting from zero. The price-clock ascends continuously and the allocation and budget are adjusted as follows.*

(1) Agents whose values are less than price-clock are removed and their allocation is frozen.

(2) *The demand of any remaining agent is the remaining budget divided by the price-clock.*

(3) *Each remaining agent clinches (and adds to their current allocation) an amount that corresponds to the largest fraction of their demand that can be satisfied when all other remaining agents are first given as much of their demand as possible.*

(4) *The budget and allocation are updated to reflect the amount clinched in the previous step.*

Proposition 3.4 (Dobzinski et al., 2008)**.** *For public-budget agents, the clinching auction always allocates all items, is ex-post IR, and is DSIC.*

Lemma 3.5 (a special case of Dobzinski et al., 2008)**.** *In single-item environment, for public-budget agents with budget w and value profile $\boldsymbol{v}$, and let k be the largest integer such that*

$$v_{(k)} \geq w \cdot k \qquad \text{where } v_{(k)} \text{ is the } k\text{-th highest value in value profile } \boldsymbol{v}.$$

Then, the agents with highest $(k-1)$ values win with same probability greater or equal to $\frac{1}{k}$ and the agent with the k-th highest value wins with the remaining probability.

We use the following approach to show that the clinching auction is an e-approximation for public-budget regular agents. We relax the feasibility constraint to an ex ante constraint and show that the optimal mechanism that sells to each agent with ex ante probability $1/n$ simply posts a price (of exactly w assuming that the budget binds) for a chance to win the item (Lemma 3.6, below). This simple form of mechanism is closely approximated by the clinching auction which sells k lotteries of $1/k$ probability (full details given subsequently). A key property is that with constant probability the budget does not bind

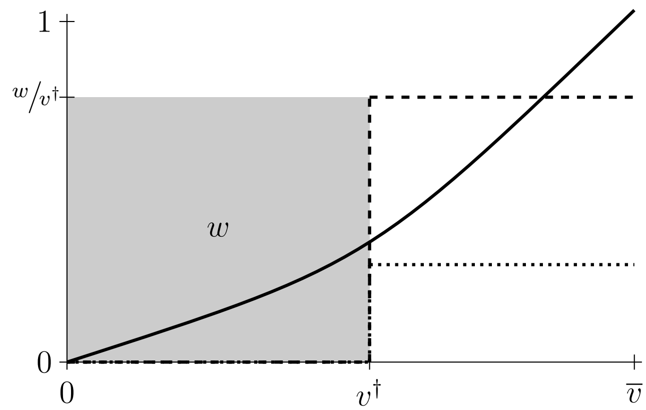


Figure 3.2. The allocation rules of the ex ante relaxation (dashed), an $1/e$-fraction of the ex ante relaxation (dotted), and the clinching auction with lotteries (solid) are depicted. The clinching auction with lotteries pointwise exceeds an $1/e$-fraction of the ex ante relaxation.

in the clinching auction with lotteries. The probability that the budget does not bind in the clinching auction with lotteries allows a lower bound on the allocation probability in the clinching auction which allows its welfare to be compared to the ex ante relaxation.

Consider the welfare-optimal auction for n agents. Since agents are symmetric, each agent will win with ex ante probability exactly $\frac{1}{n}$. We replace the feasibility constraint that ex post allocation cannot allocate more than one item (i.e. $\sum_{i\in[n]} \tilde{x}_i(\boldsymbol{v}) \leq 1$ for all $\boldsymbol{v}$) with a $\frac{1}{n}$ ex ante constraint that each agent cannot be allocated more than $\frac{1}{n}$ in expectation (i.e. $\mathbf{E}_v[x(v)] \leq {}^1\!/\!{}_n$). Ex ante optimal mechanisms for agents with public budgets were proposed and studied by Alaei et al. (2013).

Lemma 3.6 (Alaei et al., 2013)**.** *For public-budget regular i.i.d. agents with budget w, the* ex ante welfare-optimal mechanism *is either:*

(1) Budget binds: Post the price w for allocation probability $\frac{w}{v^\dagger} \leq 1$ with $v^\dagger$ set to satisfy $\frac{1}{n} = \frac{w}{v^\dagger}(1 - F(v^\dagger))$. Values $v \geq v^\dagger$ select the lottery.

(2) Allocation probability binds: Post price $v^\dagger = F^{-1}(1-\frac{1}{n})$ for allocation probability one.

We build the connection between the clinching auction and the ex ante optimal mechanism by considering the an additional auction: the clinching auction with lotteries Clinch_k which allocates k lotteries with winning probability $1/k$ per lottery, using the clinching auction framework under the same public budget. Lemma 3.7 below shows that by selecting an appropriate k, the probability that an agent with value $v^\dagger$ wins in the clinching auction with lotteries Clinch_k is at least an e fraction of the probability that the agent (with value $v^\dagger$) wins in the ex ante relaxation. See Figure 3.2.

Lemma 3.7. *For public budget i.i.d. agents, at value $v^\dagger$ defined in Lemma 3.6, there exists $k \in [n]$, such that the interim allocation of the clinching auction with lotteries $x_{Clinch_k}(v^\dagger)$ is an e-approximation of the interim allocation of the ex ante optimal mechanism $x_{PP}(v^\dagger)$, i.e., $x_{Clinch_k}(v^\dagger) \geq {}^1\!/\!_e \cdot x_{PP}(v^\dagger)$.*

Proof. Denote the notation $v_{(j:m)}$ as the j-th order statistic among m i.i.d. random variables from distribution F.

We denote the posted pricing in Lemma 3.6 as PP. Let $k_0 = 1/x_{\text{PP}}(\overline{v})$ where $x_{\text{PP}}(\overline{v})$ is the interim allocation at the highest value $\overline{v}$. By the construction of PP, $F(v^\dagger) = 1 - \frac{k_0}{n}$ and $v^\dagger \leq w \cdot k_0$ (equality holds when the budget binds in PP). Let $k = \lceil k_0 \rceil$ be the smallest integer which is greater or equal to k_0. Consider the clinching auction Clinch_k which allocates k lotteries with winning probability $1/k$ per lottery, using the clinching auction framework under public budget w.

First, fix an arbitrary agent and fix her value to be $v^\dagger$, we consider the following event $\mathcal{E}$: in Clinch_k, this agent with value $v^\dagger$ is one of the highest k valued agents and the budget does not bind. Recall that when the budget does not bind, the highest k agents in Clinch_k each receive lotteries (with allocation probability $1/k$) and pay the value of the $(k+1)$-st highest agent divided by k (i.e. $v_{(k+1:n)}/k$). The budget bids in Clinch_k if and only if $v_{(k+1:n)}/k \leq w$ and we can lower bound the lower bound the probability of the event $\mathcal{E}$ as follows,

$$\begin{aligned}
\mathbf{Pr}[\mathcal{E}] &= \mathbf{Pr}\big[(v_{(k:n-1)}/k \leq B) \wedge (v_{(k:n-1)} \leq v^\dagger)\big] \\
&= \mathbf{Pr}\big[(v_{(k:n-1)}/k \leq v^\dagger/k_0) \wedge (v_{(k:n-1)} \leq v^\dagger)\big] \\
&= \mathbf{Pr}\big[v_{(k:n-1)} \leq v^\dagger\big] \\
&= \sum_{i=0}^{k-1} \tbinom{n-1}{i} \left(\tfrac{k_0}{n}\right)^i \left(\tfrac{n-k_0}{n}\right)^{n-1-i}.
\end{aligned}$$

Above, the third line is derived from the second line using the definition of $k \geq k_0$. Denote by $x^{\mathcal{E}}$ and $x^{\bar{\mathcal{E}}}$ the allocation rule x conditioned on the events $\mathcal{E}$ and $\bar{\mathcal{E}}$, respectively. The interim allocation for Clinch_k at value $v^\dagger$ can be lower bounded as follows.

$$\begin{aligned}
x_{\text{Clinch}_k}(v^\dagger) &= x^{\mathcal{E}}_{\text{Clinch}_k}(v^\dagger) \cdot \mathbf{Pr}[\mathcal{E}] + x^{\bar{\mathcal{E}}}_{\text{Clinch}_k}(v^\dagger) \cdot \mathbf{Pr}\big[\bar{\mathcal{E}}\big] \\
&\geq x^{\mathcal{E}}_{\text{Clinch}_k}(v^\dagger) \cdot \mathbf{Pr}[\mathcal{E}] \\
&= \tfrac{k_0}{k} \cdot x_{\text{PP}}(\overline{v}) \cdot \mathbf{Pr}[\mathcal{E}] \\
&\geq \tfrac{1}{e} \cdot x_{\text{PP}}(\overline{v}).
\end{aligned}$$

The final inequality follows because the term $\frac{k_0}{k} \cdot \mathbf{Pr}[\mathcal{E}]$ achieves the minimum at $1/e$ when $k_0 = k = 1$ and n goes to infinity. $\square$

We now prove our main theorem about the approximation ratio for the clinching auction.

Theorem 3.8. *For public-budget regular i.i.d. agents, the clinching auction is an e-approximation to the welfare-optimal mechanism.*

Proof. By Lemma 3.6 the interim allocation rule of the ex ante optimal mechanism is a step function that steps at value $v^\dagger$. By Lemma 3.7, at value $v^\dagger$, the allocation rule of the clinching auction with lotteries is an e-approximation to that of the ex ante optimal mechanism. The allocation rule of the clinching auction with lotteries is monotone, so its allocation rule is an e-approximation to that of the ex ante optimal mechanism at every value. Consequently, the expected welfare of the clinching auction with lotteries is at least an e-approximation to that of the ex ante relaxation. See Figure 3.2.

Finally, Lemma 3.5 implies that for every ex post value profile, the welfare of the clinching auction is at least that of the clinching auction with lotteries. $\square$

Corollary 3.9. *For welfare maximization with public-budget regular i.i.d.agents, the optimal prior-independent approximation ratio $\Gamma(\mathbb{M}_r)$ among all revelation mechanisms is at most e.*

For public-budget regular i.i.d. agents, the all-pay auction is optimal while the clinching auction is not, since the budget binds for more value profiles in the clinching auction than in the all-pay auction. Based on this, we give a 1.03 lower bound of the approximation ratio for the clinching auction and leave the actual approximation ratio as an open problem.

Lemma 3.10. *There exists the instance of public-budget regular agents where the clinching auction is a* 1.03*-approximation of the welfare-optimal mechanism.*

Proof. Consider a simple setting: there are 2 public-budget regular agents with value drawn uniformly from $[0, \overline{v}]$ and the budget $w = 1$.

By Theorem 3.2, the all-pay auction is welfare-optimal for public-budget regular agents. The interim allocation rule of it is $x(v) = \frac{v}{\overline{v}}$ if $v \leq 2$ and $x(v) = \frac{\overline{v}+2}{2\overline{v}}$ otherwise. The expected welfare of all-pay auction is $(3\overline{v}^3 + 6\overline{v}^2 - 12\overline{v} + 8)/6\overline{v}^2$.

The interim allocation rule of the clinching auction is $x(v) = \frac{v}{\overline{v}}$ if $v \leq 1$ and $x(v) = \frac{\overline{v}+2}{2\overline{v}} - \frac{1}{2v^2}$ otherwise. The expected welfare of the clinching auction is $(3\overline{v}^3 + 6\overline{v}^2 - 3\overline{v} - 6\overline{v}\ln\overline{v} - 2)/6\overline{v}^2$.

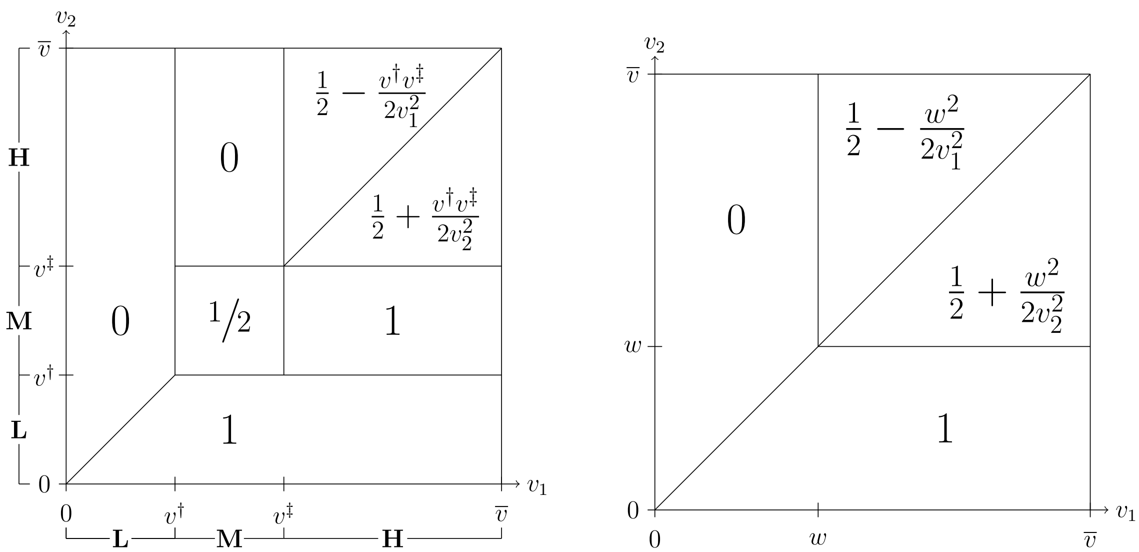
Setting $\overline{v} = 4.04$ optimizes the ratio at 1.03. $\square$

3.3. Bayesian Optimal DSIC Mechanism

In Theorem 3.2, the all-pay auction is welfare-optimal under public-budget regular distribution. Hence, applying the revelation principle to the all-pay auction, it produces a Bayesian optimal revelation mechanism. This mechanism is prior-dependent, BIC but not DSIC. In this section, we characterize the optimal DSIC mechanism for two agents with uniformly distributed values. We obtain a lower bound on its approximation ratio with the BIC optimal mechanism.

We first introduce the middle-ironed clinching auction (for two agents).

Definition 3.5. *The two-agent* middle-ironed clinching auction *is parameterized by* $v^\dagger \leq w$ *and* $v^\ddagger = 2w - v^\dagger$ *and its outcome is highest-bid-wins on values less that* $v^\dagger$*, a fair lottery on*



(a) Middle-ironed clinching auction

(b) Clinching auction

Figure 3.3. The comparison of the allocation rule $x_1(v_1, v_2)$ for the middle-ironed clinching auction and the clinching auction. In the middle-ironed clinching auction, for the values in interval **M** can be thought as "ironed", i.e. an agent receives the same outcome for any value $v \in \mathbf{M}$.

values in $[v^\dagger, v^\ddagger]$, and the clinching auction on values exceeding $v^\ddagger$; a precise formulation for two-agents is given in Figure 3.3a and a general formulation is given in Appendix A.1.

For two-agents case, the middle-ironed clinching auction allocates the item efficiently except for value profiles in **MM** (both agents with values in **M**) or **HH** (both agents with values in **H**). For the value profile in **MM**, it randomly allocates the item to one of the agent with probability $\frac{1}{2}$ with payment $\frac{v^\dagger}{2}$. For the value profile in **HH**, it allocates the item such that the budget binds for the agent with higher value and the allocation rule depends on the lower value only. Figure 3.3b depicts the allocation rule of the clinching auction for comparison. The middle-ironed clinching auction can be implemented with an ascending price via a generalization of the clinching auction that allows for price jumps which we develop in Appendix A.1 (this generalization is non-trivial).

We will show that by selecting the proper thresholds $v^\dagger$ and $v^\ddagger$, the middle-ironed clinching auction is the Bayesian optimal DSIC mechanism for two agents with uniformly distributed values. An intuition behind the optimality of the middle-ironed clinching auction is as follows: Dobzinski et al. (2008) show that for two public budget agents, the clinching auction is the only Pareto optimal (i.e. there is no outcome which is weakly better for all agents and strictly better for one agent) and DSIC auction. Moreover, after the price increases past the point where the budget binds, a differential equation governs the allocation of any DSIC mechanism. Our goal is to optimize expected welfare rather than satisfy Pareto optimality. Sacrificing welfare for lower-valued agents by ironing can delay the budget from binding and enable greater welfare from higher-valued agents. From our proof of optimality, it is sufficient to only iron one region in the middle of value space.

Theorem 3.11. *For two public-budget agents with budget w and value uniformly drawn from $[0, \overline{v}]$, Bayesian optimal DSIC mechanism is the middle-ironed clinching auction with some thresholds $v^\dagger$ and $v^\ddagger$.*

The approach of the proof is to write down our problem as a linear program (primal), assume the middle-ironed clinching auction to be the solution, and then construct the dual program with a dual solution which witnesses the optimality of the primal solution by complementary slackness. This approach is reminiscent of that of Pai and Vohra (2014) and Devanur and Weinberg (2017); however, our multi-agent DSIC constrained program presents novel challenges and for this reason we only solve the problem of two agents and uniform distributions.

We first solve a discrete version of the problem. Then, we solve the continuous version as the limit from the discrete version. Consider the value distribution with finite value space $[\overline{v}] = \{1, 2, \ldots, \overline{v}\}$ with equal probability each. We begin by writing down the optimization program for welfare maximization among all possible DSIC mechanism.

$$\begin{aligned} \sup_{(\boldsymbol{x},\boldsymbol{p})} \quad & \sum_{v_1,v_2\in[\overline{v}]} (v_1 \cdot x_1(v_1,v_2) + v_2 \cdot x_2(v_1,v_2)) \cdot \tfrac{1}{\overline{v}} \cdot \tfrac{1}{\overline{v}} \\ s.t. \quad & \\ & (\boldsymbol{x},\boldsymbol{p}) \text{ are DSIC, ex-post IR, and feasible} \\ & (\boldsymbol{x},\boldsymbol{p}) \text{ is budget balanced} \end{aligned}$$

By the characterization of dominant strategy equilibrium, we simplify this optimization program into a linear program as follows,

$$\begin{array}{lll} \max\limits_{(\boldsymbol{x},\boldsymbol{p})\geq 0} \sum\limits_{v_1,v_2\in[\overline{v}]} v_1 \cdot x(v_1,v_2) \text{ s.t.} & & \\ \overline{v} \cdot x(\overline{v}, v_2) - \sum_{t=1}^{\overline{v}} x(t, v_2) \leq w & \text{for all } v_2 \in [\overline{v}] & \text{[Budget Constraint]} \\ x(v_1,v_2) + x(v_2,v_1) \leq 1 & \text{for all } v_1, v_2 \in [\overline{v}] & \text{[Feasibility Constraint]} \\ x(v_1,v_2) \leq x(v_1+1, v_2) & \text{for all } v_1 \in [\overline{v}-1], v_2 \in [\overline{v}] & \text{[Monotonicity Constraint]} \end{array}$$

where we assume $x_1(a,b) = x_2(b,a) = x(a,b)$ for all $a, b \in [\overline{v}]$ since it is an agent-symmetric linear program. [1]

Additionally, we relax the monotonicity constraint by replacing it with $x(v_1,v_2) \leq x(\overline{v}, v_2)$ which is common for Bayesian mechanism design with public budget agents.

$$x(v_1,v_2) \leq x(\overline{v}, v_2) \quad \text{for all } v_1 \in [\overline{v}-1], v_2 \in [\overline{v}] \quad \text{[Relaxed Monotonicity Constraint]}$$

[1]Note that the program in terms of $x(a,b)$ is asymmetric.

The corresponding dual program can be written as follows. Let $\{\Lambda(v_2)\}_{v_2\in[h]}$ denote the dual variables for budget constraints; $\{\beta(v_1,v_2)\}_{v_1,v_2\in[\overline{v}]}$ denote the dual variables for feasibility constraints (for simplicity, we use both $\beta(v_1,v_2)$ and $\beta(v_2,v_1)$ to denote the same dual variable); and $\{\mu(v_1,v_2)\}_{v_1\in[\overline{v}-1],v_2\in[\overline{v}]}$ denote the dual variables for monotonicity constraints. The dual program is,

$$
\begin{aligned}
&\min_{(\boldsymbol{\Lambda},\boldsymbol{\beta},\boldsymbol{\mu})\geq 0} \sum_{v_2\in[\overline{v}]} w\cdot\Lambda(v_2)+\tfrac{1}{2}\sum_{v_1,v_2\in[\overline{v}]}\beta(v_1,v_2) \text{ s.t.} \\
&-\Lambda(v_2)+\beta(v_1,v_2)+\mu(v_1,v_2)\geq v_1 \qquad \forall v_1\in[\overline{v}-1], v_2\in[\overline{v}] \quad [x(v_1,v_2)] \\
&(\overline{v}-1)\Lambda(v_2)+\beta(\overline{v},v_2)-\textstyle\sum_{t=1}^{\overline{v}-1}\mu(t,v_2)\geq\overline{v} \quad \forall v_2\in[\overline{v}] \qquad [x(\overline{v},v_2)]
\end{aligned}
$$

The plan to solve the program is as follows. For each possible thresholds $v^\dagger, v^\ddagger$ chosen in the middle-ironed clinching auction, we first construct a solution in dual which satisfies the complementary slackness with this middle-ironed clinching auction as a solution in primal. These induced dual solutions may be infeasible. Next, we will show that there exists a pair of thresholds $v^\dagger, v^\ddagger$ which induces a feasible dual solution. This feasible dual solution witnesses the optimality of the middle-ironed clinching auction.

We will partition the dual variables into following five areas (**L**$*$, **MM**, **HH**, **MH** and **HM**) as in Figure 3.4; and construct the dual solution for them separately. We denote $\boldsymbol{\lambda}$ as the discrete derivative of the dual variable $\boldsymbol{\Lambda}$, i.e. $\lambda(v)=\Lambda(v)-\Lambda(v+1)$.

$\boldsymbol{\Lambda}$ in L:: Since the budget constraints do not bind, by complementary slackness,

$$\Lambda(v)=0 \text{ for all } v\in\mathbf{L}.$$

$\boldsymbol{\beta},\boldsymbol{\mu}$ in L$*$:: Let (v,v') be a value profile in area **L**$*$ such that $v\geq v'$. By complementary slackness on $x(v,v')$, $\beta(v,v')+\mu(v,v')-\Lambda(v')=v$ if $v<\overline{v}$; $\beta(v,v')-\sum_{t=1}^{\overline{v}-1}\mu(t,v')+$

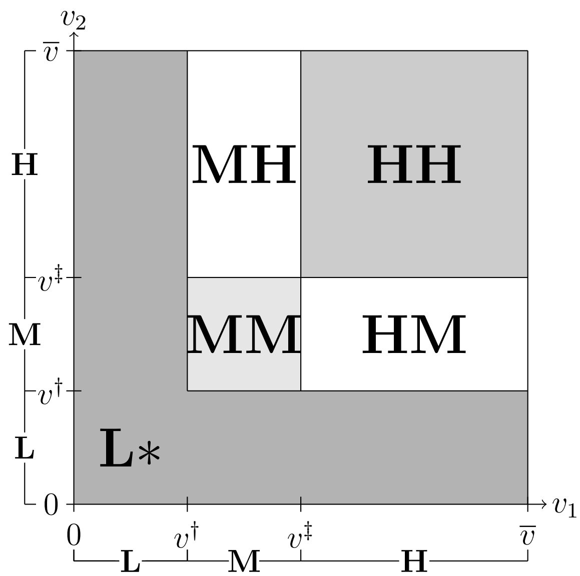


Figure 3.4. We partition the dual variables into **L**$*$ (at least one agent with value in **L**), **HH** (both agents with values in **H**), **MM** (both agents with values in **M**), **MH** and **HM** (one agent with value in **M** and the other with value in **H**) five areas.

$(\overline{v}-1)\Lambda(v') = v$ otherwise (i.e. $v = \overline{v}$). We let

$$\beta(v, v') = v \text{ and } \mu(v, v') = 0.^2$$

Since the relaxed monotonicity constraint does not bind at $x(v', v)$, i.e. $x(v', v) < x(\overline{v}, v)$, the corresponding dual variable is

$$\mu(v', v) = 0.$$

$\boldsymbol{\beta}, \boldsymbol{\mu}$ in HH:: Let (v, v') be a value profile in area **HH** such that $v \geq v'$. Since both agents win with non-zero probability, by complementary slackness on $x(v, v')$ and

[2] An intuition here is: $\boldsymbol{\mu}$ are the dual variables for the relaxed monotonicity constraint and can be thought as indicators of ironing. Though the monotonicity constraint binds, this is not because of ironing but binding allocation (i.e. $x(\cdot) \leq 1$). Therefore, we set $\boldsymbol{\mu}$ as zero.

$x(v', v)$, the corresponding dual constraints bind. Since the relaxed monotonicity constraint does not bind at $x(v', v)$, the monotonicity dual variable is

$$\mu(v', v) = 0.$$

The binding dual constraint of $x(v', v)$ is $\beta(v', v) - \Lambda(v) + \mu(v', v) = v'$. Hence,

$$\beta(v, v') = \beta(v', v) = v' + \Lambda(v).^3$$

The binding dual constraint of $x(v, v')$ is $\beta(v, v') - \Lambda(v') + \mu(v, v') = v$. Note the relaxed monotonicity constraint is tight for (v, v'). Hence,

$$\mu(v, v') = v - v' + \Lambda(v') - \Lambda(v).$$

Here we write $\boldsymbol{\beta}, \boldsymbol{\mu}$ as terms of $\boldsymbol{\Lambda}$. In the next paragraph, we will solve for $\boldsymbol{\Lambda}$.

$\boldsymbol{\Lambda}$ in H:: Let $v \in \mathbf{H}$. Consider the binding dual constraint of $x(\overline{v}, v)$, $(\overline{v} - 1)\Lambda(v) + \beta(\overline{v}, v) - \sum_{t=1}^{\overline{v}-1} \mu(t, v) = \overline{v}$. Notice that by complementary slackness, $\mu(t, v) = 0$ for all $t \leq v$. Plugging $\boldsymbol{\beta}$ and $\boldsymbol{\mu}$ as terms of $\boldsymbol{\Lambda}$ into the these dual constraints of $x(\overline{v}, v)$, we can solve for $\boldsymbol{\Lambda}$ as

$$\lambda(v) = \frac{\overline{v} - v}{v} \text{ for all } v \in \mathbf{H} \text{ and } \Lambda(\overline{v}) = 0.^4$$

$\boldsymbol{\beta}, \boldsymbol{\mu}$ in MM and $\boldsymbol{\Lambda}$ in M:: Let (v, v') be a value profile in area **MM** such that $v \geq v'$. Since the relaxed monotonicity constraints do not bind for either $x(v, v')$ or

[3] Recall that $\beta(v, v')$ and $\beta(v', v)$ denote the same dual variable.

[4] Recall that $\boldsymbol{\lambda}$ is the discrete derivative of dual variables $\boldsymbol{\Lambda}$, so $\Lambda(v) = \sum_{t=v}^{\overline{v}-1} \lambda(v)$.

$x(v', v)$, the corresponding dual variables are

$$\mu(v, v') = \mu(v', v) = 0.$$

The binding dual constraints of $x(v, v')$ implies $\beta(v, v') = v' + \Lambda(v)$. On the other hand, the binding dual constraints of $x(v', v)$ implies $\beta(v', v) = v + \Lambda(v')$. Recall that $\beta(v, v')$ and $\beta(v', v)$ denote the same variable, hence,

$$\lambda(v) = -1 \text{ for all } v \in \mathbf{M} \setminus \{v^{\ddagger} - 1\},$$ [5]

$$\beta(v, v') = \Lambda(v^{\ddagger}) + \lambda(v^{\ddagger} - 1) + v + v' - v^{\ddagger}.$$

$\boldsymbol{\beta}, \boldsymbol{\mu}$ in MH and HM:: Let (v, v') be a value profile in area **HM** such that $v > v'$. With the similar argument for region **HH**,

$$\mu(v', v) = 0 \text{ and } \mu(v, v') = v - v' + \Lambda(v') - \Lambda(v),$$

$$\beta(v, v') = v' + \Lambda(v) \text{ if } v < \overline{v}.$$

Plugging the above expressions for $\boldsymbol{\mu}$ into the binding dual constraint of $x(\overline{v}, v')$,

$$\beta(\overline{v}, v') = (\overline{v} - 1)(v^{\ddagger} - v') + 1 + (v^{\ddagger} - 1)\lambda(v^{\ddagger} - 1).$$

[5]Complementary slackness does not pin down $\lambda(v^{\ddagger} - 1)$. We leave it as a variable and identify it later when we choose the thresholds $v^{\dagger}$, $v^{\ddagger}$ to ensure that the dual solution is feasible.

With the analysis above, we construct the following dual solution which satisfies complementary slackness with the middle-ironed clinching auction as a solution in primal,

$$
\Lambda(v_2) = \begin{cases} 0 & \text{if } v_2 < v^\dagger \\ \sum_{k=v^\ddagger}^{\overline{v}-1} \frac{\overline{v}-k}{k} + v_2 - v^\ddagger + 1 + \lambda(v^\ddagger - 1) & \text{if } v^\dagger \le v_2 < v^\ddagger \\ \sum_{k=v_2}^{\overline{v}-1} \frac{\overline{v}-k}{k} & \text{if } v_2 \ge v^\ddagger \end{cases}
$$

$$
\beta(v_1, v_2) = \begin{cases} v_1 \quad \text{if } v_1 \ge v_2,\ v_2 < v^\dagger \\ \sum_{k=v^\ddagger}^{\overline{v}-1} \frac{\overline{v}-k}{k} + v_1 + v_2 - v^\ddagger + 1 + \lambda(v^\ddagger - 1) \\ \qquad \text{if } v_1 \ge v_2,\ v^\dagger \le v_2 < v^\ddagger,\ v_1 < v^\ddagger \\ \sum_{k=v_1}^{\overline{v}-1} \frac{\overline{v}-k}{k} + v_2 \\ \qquad \text{if } v_1 \ge v_2,\ v^\dagger \le v_2 < v^\ddagger,\ v^\ddagger \le v_1 \le \overline{v} - 1 \\ (\overline{v} - 1)(v^\ddagger - v_2) + 1 + (v^\ddagger - 1)\lambda(v^\ddagger - 1) \\ \qquad \text{if } v_1 \ge v_2,\ v^\dagger \le v_2 < v^\ddagger,\ v_1 = \overline{v} \\ \sum_{k=v_1}^{\overline{v}-1} \frac{\overline{v}-k}{k} + v_2 \quad \text{if } v_1 \ge v_2,\ v^\ddagger \le v_2 \end{cases} \tag{3.1}
$$

$$
\mu(v_1, v_2) = \begin{cases} 0 & \text{if } v_2 < v^\dagger \\ 0 & \text{if } v^\dagger \le v_2 < v^\ddagger, v_1 < v^\ddagger \\ v_1 - v^\ddagger + \sum_{k=v^\ddagger}^{v_1-1} \frac{\overline{v}-k}{k} + 1 + \lambda(v^\ddagger - 1) & \text{if } v^\dagger \le v_2 < v^\ddagger, v_1 \ge v^\ddagger \\ 0 & \text{if } v_2 \ge v^\ddagger, v_1 \le v_2 \\ v_1 - v_2 + \sum_{k=v_2}^{v_1-1} \frac{\overline{v}-k}{k} & \text{if } v_2 \ge v^\ddagger, v_1 > v_2 \end{cases}
$$

Lemma 3.12. *For the middle-ironed clinching auction with arbitrary thresholds $v^\dagger$ and $v^\ddagger$, the dual solution* (3.1) *satisfies the complementary slackness.*

Proof. The complementary slackness is directly implied by the construction. $\square$

Though the this dual solution satisfies the complementary slackness, it may be infeasible. Therefore, we argue that there exists some thresholds $v^\dagger$, $v^\ddagger$ and $\lambda(v^\ddagger - 1)$ under which the dual solution is feasible.

Lemma 3.13. *There exists $v^\dagger, v^\ddagger$ and $\lambda(v^\ddagger - 1)$ such that the constructed dual solution* (3.1) *is feasible.*

Proof. We define function $Z(v) = 2v - 2w - 2 - \sum_{k=v}^{\overline{v}-1} \frac{\overline{v}-k}{k}$ to simplify the argument. Notice that $\Lambda(v) = \sum_{k=v}^{\overline{v}-1} \frac{\overline{v}-k}{k}$ in the dual solution (3.1) if $v \in \mathbf{H}$.

Due to complementary slackness, all dual constraints corresponding to some $x(v, v') > 0$ bind, so they are satisfied automatically. Hence, to ensure the constructed dual solution is feasible, there remain four groups of constraints which need to be satisfied. For each group of constraints, there is a "pivotal" constraint such that if it is satisfied, all constraints in that group is satisfied. We list these four groups of constraints and "pivotal" constraint for each group below,

All dual constraints of $x(v, v')$ where $v \in \mathbf{L}$ and $v' \in \mathbf{M}$:: The pivotal constraint is the dual constraint of $x(v^\dagger - 1, 1)$, which can be simplified as

$$\lambda(v^\ddagger - 1) \leq Z(v^\ddagger).$$

All dual constraints of $x(v, v')$ where $v \in \mathbf{L}$ and $v' \in \mathbf{H}$:: The pivotal constraint is the dual constraint of $x(v^\dagger - 1, v^\ddagger)$, which can be simplified as

$$-1 \leq Z(v^\ddagger).$$

All dual constraints of $x(v, \overline{v})$ where $v \in$ M:: The pivotal constraint is the dual constraint of $x(v^\ddagger - 1, \overline{v})$, which can be simplified as

$$\lambda(v^\ddagger - 1) \leq \frac{\overline{v}}{v^\ddagger - 1} - 1.$$

$\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{\beta} \geq 0$:: The pivotal constraint is $\Lambda(v^\dagger) \geq 0$, which can be simplified as

$$\lambda(v^\ddagger - 1) \geq Z(v^\ddagger) - 1.$$

We now show how to relate $v^\dagger$, $v^\ddagger$ and $\lambda(v^\ddagger - 1)$ to satisfy the four inequalities identified above.

Notice that when $v^\dagger = 1$ and $v^\ddagger = 2w + 1$, the interval **L** becomes empty. In that case, the first and second groups of constraints disappear. The combination of these four inequalities is equivalent to

i. $v^\ddagger = 2w + 1$ and $Z(v^\ddagger) \leq \frac{\overline{v}}{v^\ddagger - 1}$; or

ii. $-1 \leq Z(v^\ddagger) \leq \frac{\overline{v}}{v^\ddagger - 1}$.

Without loss of generality, we assume that Condition (i) does not hold and then argue that Condition (ii) holds in this case.

The construction of $Z(\cdot)$ implies the following two facts,

(a) if $Z(v) < -1$, then $Z(v+1) < \frac{\overline{v}}{v}$;

(b) if $Z(v) > \frac{\overline{v}}{v-1}$, then $Z(v-1) > -1$.

If we think of interval $(-1, \frac{\overline{v}}{v-1})$ as a “window”, these two facts say that if a point $Z(v)$ is on the left hand side of this window (i.e. $Z(v) < -1$) , then the next point $Z(v+1)$ cannot jump to the right hand side of the window (i.e. it is either in the window or still on the left hand side of the window) and vice versa. Notice that $Z(w+1) = -\sum_{k=w+1}^{\overline{v}-1} \frac{\overline{v}-k}{k} \le 0 < \frac{\overline{v}}{w}$ and our assumption (i.e. Condition (i) does not hold) implies $Z(2w+1) > \frac{\overline{v}}{2w} > -1$. Thus, there exists $v^\ddagger$ such that $-1 \le Z(v^\ddagger) \le \frac{\overline{v}}{v^\ddagger - 1}$, i.e. Condition (ii) holds. $\square$

The construction of the dual solution which satisfies feasibility and complementary slackness witnesses the optimality of the middle-ironed clinching auction. We offer the following discrete version of Theorem 3.11.

Theorem 3.14. *For two public-budget agents with value uniformly distributed from $[\overline{v}]$, the Bayesian optimal DSIC mechanism is the middle-ironed clinching auction for some thresholds $v^\dagger$ and $v^\ddagger$.*

We now focus on continuous uniform distribution with value space $[0, \overline{v}]$. Again, we write the problem as an optimization program as follows,

$$
\begin{aligned}
&\max_{(x,p)\ge 0} \int_0^{\overline{v}} \int_0^{\overline{v}} v_1 \cdot x(v_1, v_2) dv_2 dv_1 \text{ s.t.} && \\
&\overline{v} \cdot x(\overline{v}, v_2) - \textstyle\int_0^{\overline{v}} x(t, v_2) dt \le w && \forall v_2 \in [0, \overline{v}] \quad \text{[Budget Constraint]} \\
&x(v_1, v_2) + x(v_2, v_1) \le 1 && \forall v_1, v_2 \in [0, \overline{v}] \quad \text{[Feasibility Constraint]} \\
&x(v_1, v_2) \le x(\overline{v}, v_2) && \forall v_1, v_2 \in [0, \overline{v}] \quad \text{[Relaxed Monotonicity Constraint]}
\end{aligned}
$$

Proof of Theorem 3.11. Discretize the value space $[0, \overline{v}]$ into $\{\epsilon, 2\epsilon, \ldots, m\epsilon\}$ where $m\epsilon = \overline{v}$ with density $\frac{1}{m}$ each. Define $\mathcal{X}_\epsilon$ to be the class of all possible DSIC, ex-post IR, budget balanced allocations such that each value $v \in [(k-1)\epsilon, k\epsilon)$ must be ironed for all $k = 1, \ldots, \overline{v}$. By the construction of $\mathcal{X}_\epsilon$, the allocation function x^ϵ in Theorem 3.14 indeed solves $\max_{x \in \mathcal{X}_\epsilon} \int_0^{\overline{v}} \int_0^{\overline{v}} v_1 \cdot x(v_1, v_2) dv_2 dv_1$ after rescaling both value space and budget by $\frac{1}{\epsilon}$.

Let $\mathcal{X}$ be the class of all possible DSIC, ex-post IR, budget balanced allocations. Notice that $\mathcal{X}_\epsilon$ converges to $\mathcal{X}$ pointwise and that both are compact subsets of the L_1 space defined by uniform measure. The operator $T(x) = \int_0^{\overline{v}} \int_0^{\overline{v}} v_1 \cdot x(v_1, v_2) dv_2 dv_1$ is a bounded linear operator from the L_1 space of allocation function to $\mathbb{R}$. Therefore, T achieves its maximum on each set $\mathcal{X}^\epsilon$ and $\mathcal{X}$.

The pointwise convergence ensures that

$$\lim_{\epsilon \to 0} \max_{x \in \mathcal{X}^\epsilon} \int_0^{\overline{v}} \int_0^{\overline{v}} v_1 \cdot x(v_1, v_2) dv_2 dv_1 = \max_{x \in \mathcal{X}} \int_0^{\overline{v}} \int_0^{\overline{v}} v_1 \cdot x(v_1, v_2) dv_2 dv_1$$

Since $T(x)$ is a bounded linear operator and $\{x^\epsilon\}$ has a pointwise limit,

$$\lim_{\epsilon \to 0} x^\epsilon \in \{\text{argmax}_{x \in \mathcal{X}} \int_0^{\overline{v}} \int_0^{\overline{v}} v_1 \cdot x(v_1, v_2) dv_2 dv_1\}.$$

Thus, we see that Theorem 3.11 holds. □

Based on Theorem 3.11, the performance of the welfare-optimal DSIC and BIC mechanisms can be compared.

Lemma 3.15. *There exists the instance of public-budget regular agents where the welfare-optimal DSIC mechanism is a* 1.013*-approximation to the welfare-optimal BIC mechanism.*

Proof. Consider two agents with values drawn uniformly from $[0, \overline{v}]$ where $\overline{v} \geq 5.5$ and the budget $w = 1$. By Theorem 3.11, the welfare-optimal DSIC mechanism in this case is the middle-ironed clinching auction with $v^\dagger = 0$ and $v^\ddagger = 2$. The welfare-optimal BIC mechanism is the all-pay auction (applying the revelation principle). By computing the welfare for both mechanisms under this distribution, and setting $\overline{v} = 5.5$, the ratio is optimized as 1.013. □

3.4. Revelation Gap for Welfare Maximization

Theorem 3.16. *For public-budget regular i.i.d.agents, the revelation gap for welfare maximization is at most e. Specifically, this upper bound considers prior-independent DSIC, ex-post IR mechanisms.*

Proof. This upper bound is given by considering the clinching auction which is a prior-independent DSIC and ex-post IR mechanism. Theorem 3.8 says that the clinching auction is an e-approximation to the welfare-optimal mechanism for public-budget regular agents. Thus, the revelation gap is at most e. □

For the lower bound, we use the result in section 3.3 where we solve the welfare-optimal DSIC mechanism for two agent with uniformly distributed values. Note that for two-agent environments, the DSIC ex-post IR constraints are equivalent to prior-independent BIC and IIR constraints.[6] With more than two agents, this equivalence does not generally hold.

Lemma 3.17. *For two i.i.d.agents, a mechanism is Bayesian incentive compatible and interim individual rational for all i.i.d.distributions if and only if it is dominant strategy incentive compatible and ex-post individually rational.*

Proof. The direction that DSIC implies BIC for all i.i.d.distribution is trivial by the definition. To show the other direction, for arbitrary value v, consider the distribution which puts the whole mass on v. These distributions break the interim constraints in

[6]Prior-independent BIC and IIR mechanisms are the mechanisms which are BIC and IIR for all i.i.d.distributions. This property is stronger than BIC (for a single distribution) but generally weaker than DSIC.

BIC into the ex-post constraints in DSIC for every value profiles. Hence, BIC for all i.i.d.distribution implies DSIC for two agents setting. □

Theorem 3.18. *For public-budget regular i.i.d.agents, the revelation gap for welfare maximization is at least* 1.013.

Proof. This lower bound is given by considering the all-pay auction and the middle-ironed clinching auction.

As the characterization in Section 3.1, the all-pay auction is a prior-independent mechanism. Theorem 3.2 says that the all-pay auction is welfare-optimal for public budget regular agents. Hence, the prior-independent approximation of mechanisms is 1.

Next, we show that the prior-independent approximation of Bayesian incentive compatible mechanisms is at least 1.013. Theorem 3.11 says that the middle-ironed clinching auction is Bayesian optimal DSIC mechanism for two agents with values drawn uniformly from $[0, \overline{v}]$. Since for two agents case, the DSIC property is equivalent to the BIC for all i.i.d.distribution property, Lemma 3.15 suggests that the prior-independent approximation of incentive compatible mechanisms is at least 1.013.

Thus, the revelation gap for welfare maximization is at least 1.013. □

3.5. Welfare Approximation for Irregular Distribution

In this section, we analyze the prior-independent approximation ratio of the all-pay auction and the clinching auction for public budget agents without public-budget regularity assumption.

The main technique we use is the following lemma which relaxes the budget constraint to another constraint which upper bounds the wining probability of the highest value, i.e.$x(\overline{v})$ where $\overline{v}$ is the highest value in the support of the distribution.

Lemma 3.19. *Given any interim constraint $x^\star$ and budget w, let $v^\dagger$ be the value where the budget binds in $x^\star$ after ironing from $v^\dagger$ to $\overline{v}$, i.e.$v^\dagger \cdot z^\star(v^\dagger) - \int_0^{v^\dagger} x^\star(t)dt = w$ where $z^\star(v^\dagger) = \frac{1}{1-F(v^\dagger)} \int_{v^\dagger}^{\overline{v}} x^\star(t)dF(t)$, the averaging wining probability for value beyond $v^\dagger$ in allocation $x^\star$. Any interim feasible and budget balanced allocation x satisfies*

$$x(\overline{v}) \leq 2z^\star(v^\dagger).$$

Proof. Recall that $v^\dagger$ is the value where the budget binds in $x^\star$ after ironing from $v^\dagger$ to $\overline{v}$. Thus,

$$w = v^\dagger \cdot z^\star(v^\dagger) - \int_0^{v^\dagger} x^\star(t)dt \leq v^\dagger \cdot z^\star(v^\dagger).$$

On the other hand, suppose x is budget balance,

$$w \geq \overline{v} \cdot x(\overline{v}) - \int_0^{\overline{v}} x(t)dt \geq v^\dagger \cdot (x(\overline{v}) - x(v^\dagger))$$

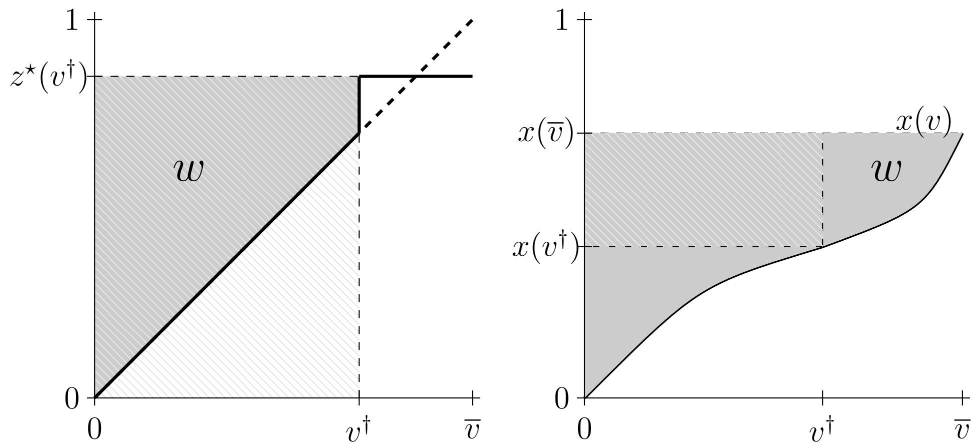
Suppose x is interim feasible,

$$x(v^\dagger) \leq \frac{1}{1-F(v^\dagger)} \int_{v^\dagger}^{\overline{v}} x(t)dt \leq \frac{1}{1-F(v^\dagger)} \int_{v^\dagger}^{\overline{v}} x^\star(t)dt = z^\star(v^\dagger).$$

Combine the inequalities above,

$$x(\overline{v}) \leq x(v^\dagger) + z^\star(v^\dagger) \leq 2z^\star(v^\dagger).$$

□



(a) $w \leq v^\dagger \cdot z^\star(v^\dagger)$ (b) $w \geq v^\dagger \cdot (x(\overline{v}) - x(v^\dagger))$

Figure 3.5. Proofs by picture of the upper bound and lower bound on budget w.

The All-pay Auction

First, we discuss the performance of all-pay auction for the irregular distribution. Pai and Vohra (2014) show that the welfare-optimal interim allocation is both ironing top interval and perhaps ironing some other intervals in the middle. It turns out that even though the all-pay auction only irons the top interval, its welfare only suffers a modest loss.

Theorem 3.20. *For public-budget i.i.d.agents, the all-pay auction is a 2-approximation to the welfare-optimal mechanism.*

Proof. Applying Lemma 3.19, we relax the budget constraint to the constraint that $x(\overline{v}) \leq 2z^\star(v^\dagger)$.

Denote x^0 as the welfare-optimal interim feasible and budget balanced allocation and x as the welfare-optimal interim feasible allocation under the relaxed constraint, then $\mathbf{welfare}[x^0] \leq \mathbf{welfare}[x]$.

Since x maximizes the welfare under interim feasibility constraint $x^\star$, it allocates as by $x^\star$ except that values in $[v^\ddagger, \overline{v}]$ are ironed. The threshold $v^\ddagger$ is selected such that $x(\overline{v}) = 2z^\star(v^\dagger)$. By definitions of $v^\dagger$ and $v^\ddagger$, we know $v^\dagger \leq v^\ddagger$. Consider x for values below and beyond $v^\dagger$ separately. For value below $v^\dagger$, the expected welfare $\int_0^{v^\dagger} v \cdot x(v)dF(v) = \int_0^{v^\dagger} v \cdot x^\star(v)dF(v)$. For value beyond $v^\dagger$, the expected welfare $\int_{v^\dagger}^{\overline{v}} v \cdot x(v)dF(v) \leq \int_{v^\dagger}^{\overline{v}} v \cdot 2z^\star(v^\dagger)dF(v)$.

Notice that $v^\dagger$ coincides with the threshold in the all-pay auction, and the all-pay auction allocates as by $x^\star$ except value beyond $v^\dagger$ win with probability $z^\star(v^\dagger)$. Thus, **welfare**$[x] \leq 2 \cdot$ **welfare**[All-pay]. $\square$

In fact, the 2-approximation bound is tight.

Lemma 3.21. *There exists the instance where the welfare of the all-pay auction is half of welfare-optimal mechanism.*

Proof. Consider the following single-item instance with budget $w = 1$. There are $N + 1$ agents with valuation distribution

$$v = \begin{cases} N - \epsilon & \text{w.p. } \frac{1}{N+1}, \\ N & \text{w.p. } \frac{N-1}{N+1}, \\ N^3 & \text{w.p. } \frac{1}{N+1}. \end{cases}$$

In the all-pay auction, the interim allocation rule irons values N and N^3,

$$x(v) = \begin{cases} \delta & \text{if } v = N - \epsilon, \\ \frac{1-\delta}{N} & \text{if } v = N \text{ or } N^3, \end{cases} \qquad \text{where } \delta = (\tfrac{1}{(N+1)})^{N+1} \to 0.$$

where the expected welfare is roughly $N + 1$.

Notice that in the all-pay auction, the mechanism uses almost all budget to distinguish values $N-\epsilon$ and N whose contribution to the expected welfare is almost the same. Therefore, consider the auction which irons values $N-\epsilon$ and N, and moves some winning probability from value N^3 to values $N-\epsilon, N$ as follows,

$$x(v) = \begin{cases} \frac{N-1}{N(N+1)} & \text{if } v = N-\epsilon \text{ or } N, \\ \frac{2}{N+1} & \text{if } v = N^3. \end{cases}$$

where the expected welfare is roughly $2N+1$.

Let $N \to \infty$ and $\epsilon \to 0$, the expected welfare from the all-pay auction is exactly half of the expected welfare from the optimal auction. $\square$

The Clinching Auction

To analyze the welfare approximation of the clinching auction for irregular distributions, we follow almost the same argument as for regular distributions. The only difference in the argument is that the ex ante welfare-optimal mechanism may not be a simple posted price for a probabilistic allocation. However, by Lemma 3.19, such a posted pricing is still a 2-approximation.

Lemma 3.22. *For public-budget i.i.d.agents, the posted pricing described in Lemma 3.6 is a 2-approximation to the ex ante welfare-optimal mechanism.*

Proof. Consider the interim constraint $x^{\star}(v) = 1$ if $F(v) \geq 1/n$ and $x^{\star}(v) = 0$ if $F(v) < 1/n$. Applying the similar argument in Theorem 3.20 with Lemma 3.19, the lemma holds. $\square$

The following corollary is combines Lemma 3.22 with Theorem 3.8.

Corollary 3.23. *For public-budget i.i.d. agents, the clinching auction is a* $2e$*-approximation to the welfare-optimal mechanism.*

CHAPTER 4

Revenue Maximization for A Linear Agent with A Single Sample Access

In this chapter, we consider prior-independent mechanism design for a linear agent with a single sample access. Since we only focus on single-agent problem, we drop the agent's index in the subscript of all notations.

In Section 4.1, we discuss how to extend the basic prior-independent mechanism design problem to the one with a single sample access, and introduce some necessary technique for the later analysis. In Section 4.2, we introduce the sample-bid mechanism – a non-revelation mechanism and discuss some basic properties. We upper-bound the prior-independent approximation ratio of the sample-bid mechanism in Sections 4.3 and 4.4 within the class of MHR, regular distributions respectively. In Section 4.5, we show a lower bound of the optimal prior-independent approximation ratio. In Section 4.6, we focus on the sample-based pricing – a revelation mechanism, and discuss its prior-independent approximation ratio. Combining all results from Sections 4.3 to 4.6, we establish the non-trivial revelation gaps in revenue maximization in this setting.

4.1. Preliminaries: Single-item Auction with A Single Sample Access

When there is only a single agent, there is no prior-independent mechanism with finite prior-independent approximation ratio for revenue maximization, unless the class of distributions DISTS is trivial (e.g., DISTS is a singleton). To address this issue, we allow the mechanism to access a sample from the valuation distribution.

Recall that in the classic prior-independent mechanism design defined in Chapter 2, the seller does not know the valuation distribution F of the agent. However, in this chapter, we assume that the seller has a single sample s drawn from F. The agent knows the valuation distribution F but does not observe the sample s, and the value v of the agent is independent of the sample s. A mechanism $\mathcal{M} = (\tilde{x}, \tilde{p})$ includes an allocation rule $\tilde{x} : \mathbb{R} \times \mathbb{R} \to [0, 1]$ mapping from the agent's bid b and the sample s to the allocation probability of the item; and a payment rule $\tilde{p} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ mapping from the agent's bid b and the sample s to the payment charged from the agent. Let $\tilde{x}(b, F) = \mathbf{E}_{s\sim F}[\tilde{x}(b, s)]$, $\tilde{p}(b, F) = \mathbf{E}_{s\sim F}[\tilde{p}(b, s)]$ be the expected allocation and payment over the randomness of the sample s drawn from distribution F. The seller first announce the mechanism $\mathcal{M} = (\tilde{x}, \tilde{p})$ to the buyer, and then the sample s and value v are realized from distribution F. The agent report a bid b based on her private value v, and the seller implements the mechanism $\mathcal{M}$ with input b and sample s. We assume that the seller has full commitment power on implementing the mechanism.

Given a mechanism $(\tilde{x}, \tilde{p})$ and distribution F, the *best response* of the agent is $b(\cdot, F) : \mathbb{R} \to \mathbb{R}$ which maximizes her expected utility, i.e., for every value v, $b(v, F) \in \operatorname{argmax}_b v \cdot \tilde{x}(b, F) - \tilde{p}(b, F)$.[1] A mechanism $(\tilde{x}, \tilde{p})$ is *incentive compatible (IC)* if reporting the agent's value truthfully is her best response, i.e., $b(v, F) = v$ for all v and F. A mechanism $(\tilde{x}, \tilde{p})$

[1]When there are multiple bids maximizing the utility of the agent, we allow the agent to choose any bid maximizing her utility. The revenue guarantee we obtained in this paper holds even when the agent can break tie and choose the bid minimizing the revenue of the seller.

is *individual rational (IR)* if the agent's utility under her best response is non-negative, i.e, $\max_b v \cdot \tilde{x}(b, F) - \tilde{p}(b, F) \geq 0$ for all v and F.[2]

For any mechanism $(\tilde{x}, \tilde{p})$, let $x(v, F, s) = \tilde{x}(b(v, F), s)$ be the interim allocation of value v given distribution F and sample s when the agent follows her best response, and let $p(v, F, s) = \tilde{x}(b(v, F), s)$ be the interim payment. Moreover, denote $x(v, F) = \mathbf{E}_{s\sim F}[x(v, F, s)]$ and $p(v, F) = \mathbf{E}_{s\sim F}[p(v, F, s)]$ as the expected interim allocation and payment. We often omit F in the notation if it is clear from the context.

The revenue $\mathbf{Rev}_F[\mathcal{M}]$ of a mechanism $\mathcal{M} = (x, p)$ on distribution F is the expected payment when the agent plays her best response, i.e., $\mathbf{E}_{v\sim F}[p(v, F)]$. We evaluate mechanisms by the prior-independent approximation ratio.

Definition 4.1 (a special case of Definition 2.7)**.** *The* prior-independent approximation ratio *of a mechanism $\mathcal{M}$ over a class of distributions DISTS is defined as*

$$\mathit{\Gamma}(\mathcal{M}, \mathit{DISTS}) \triangleq \max_{F \in \mathit{DISTS}} \frac{\mathbf{Rev}_F[\mathrm{OPT}_F]}{\mathbf{Rev}_F[\mathcal{M}]}$$

where $\mathbf{Rev}_F[\mathrm{OPT}_F] \triangleq \max_p \, (1 - F(p))\, p$ *is the optimal revenue for distribution F (cf. Myerson, 1981).*

Revenue Curve. For any distribution F, let $q(v, F) = 1 - F(v)$ be the quantile for the distribution, and $v(q, F)$ be the value v such that $q = 1 - F(v)$. Here we introduce the revenue curve in quantile space (cf. Bulow and Roberts, 1989), which is a useful tool in the revenue analysis.

[2]Note that the utility of the agent can be negative for some realization of the sample s, but in expectation it must be non-negative.

Definition 4.2. *For any valuation distribution F, the revenue curve $R(q, F)$ of the agent is a mapping from any $q \in [0, 1]$ to the optimal revenue from an agent with value drawn from F subject to the constraint that the item is allocated with ex ante probability q.*

In the later analysis in the paper, when F is clear from the context, we omit it in the notation and only use $R(q)$ to represent the revenue curve and $q(v)$ to represent the quantile of value v. Let $\phi(v) = v - \frac{1-F(v)}{f(v)}$ be the virtual value of the agent.

Definition 4.3. *An valuation distribution F is regular if the virtual value of the agent is weakly increasing.*

Theorem 4.1 (Myerson, 1981)**.** *A distribution F is regular if and only if the corresponding revenue curve $R(q, F)$ is concave.*

Finally, we define the monopoly reserve and monopoly quantile of the agent given the revenue curve R.

Definition 4.4. *The monopoly quantile of the agent is $\hat{q}^\star = \operatorname{argmax}_q R(q)$,*[3] *and the monopoly reserve of the agent is $\hat{v}^\star = R(\hat{q}^\star)/\hat{q}^\star$.*

4.2. The Sample-bid Mechanism

In this section, we introduce the main mechanism considered in this paper, the sample-bid mechanism.

Definition 4.5 (sample-bid mechanism)**.** *Given parameter α and sample s, the* sample-bid mechanism *solicits a non-negative bid $b \geq 0$, charges the agent $\alpha \cdot \min\{b, s\}$, and allocates the item to the agent if $b \geq s$.*

[3]In this paper, we break tie in favor of smaller quantile. Note that all the results are not affected by the tie breaking rule.

In the sample-bid mechanism, the agent reports her bid without knowing the realization of the sample. From her perspective, the utility $u(v, b, F)$ for her who has value v, reports bid b, and competes with sample $s \sim F$ is

$$u(v, b, F) = v \cdot \underbrace{F(b)}_{\mathbf{Pr}_{s\sim F}[s\leq b]} - \underbrace{\alpha b \cdot (1 - F(b))}_{\text{payment when } s \geq b} - \underbrace{\alpha \int_{\underline{v}}^{\max\{b,\underline{v}\}} t dF(t)}_{\text{payment when } s \leq b}$$

Note that reporting bid equal to zero, the utility of agent is zero. Thus, sample-bid mechanismis individually rational.

Lemma 4.2. *The sample-bid mechanism is individually rational.*

On the other hand, reporting bid equal to agent's value is not the best response in general. We provide a characterization of agent's optimal bid as follows.

Lemma 4.3. *In the sample-bid mechanism, given any parameter α and distribution F, the optimal bid $b(v, F)$ for the agent with value v satisfies the constraint that*

$$v = \alpha \cdot \frac{1 - F(b(v, F))}{f(b(v, F))}, \tag{4.1}$$

or $b(v, F) \in \{0, \infty\}$. Ties are broken according to the utility of the agent.

Proof. The agent's utility from reporting bid b is

$$u(v, b, F) = v \cdot F(b) - \alpha b(1 - F(b)) - \alpha \int_{\underline{v}}^{\max\{b,\underline{v}\}} t dF(t)$$

Consider the first order condition with respect to bid b, if the optimal bid is obtained in the interior, we have

$$f(b)\left(v - \alpha \cdot \frac{1 - F(b)}{f(b)}\right) = 0$$

as a necessary condition for the optimality of the bid b. Otherwise, the optimal bid is obtained on the boundary, where $b(v, F) \in \{0, \infty\}$. $\square$

Note that in Lemma 4.3 there might exist multiple bids b that satisfies the constraint (4.1). In that case, the agent chooses the bid which satisfies (4.1) and maximizes her utility. Another observation (Lemma 4.4) of the sample-bid mechanism is that the expected revenue of the seller scales linearly with the valuation distribution. Since the optimal revenue scales linearly with the valuation distribution as well, to analyze the prior-independent approximation ratio of the sample-bid mechanism, we can focus on the valuation distributions such that the optimal revenue is normalized to 1.

Lemma 4.4. *Denote by r the revenue of the sample-bid mechanism with any parameter α and any valuation distribution $F^\dagger$. For any $\rho > 0$ and distribution $F^\ddagger$ such that $F^\ddagger$ is $F^\dagger$ scaled by ρ, i.e., $F^\dagger(v) = F^\ddagger(\rho v)$ for all v, the revenue of the sample-bid mechanism with parameter α and distribution $F^\ddagger$ is ρr.*

Proof. First we show that for any value v, the bid of value v given distribution $F^\dagger$ is equivalent to the bid of value ρv given distribution $F^\ddagger$ scaled by ρ. The reason is that $F^\dagger(v) = F^\ddagger(\rho v)$ and $f^\dagger(v) = \rho f^\ddagger(\rho v)$. Therefore, by Lemma 4.3, the first order condition implies that the optimal bid satisfies $b(\rho v, F^\ddagger) = \rho \cdot b(v, F^\dagger)$. Moreover, the payment satisfies

$$\begin{aligned}\tilde{p}(\rho b, F^{\ddagger}) &= \alpha\rho b \cdot (1 - F^{\ddagger}(\rho b)) + \alpha \int_0^{\rho b} t \mathrm{d}F^{\ddagger}(t) \\ &= \rho(\alpha b \cdot (1 - F^{\dagger}(b) + \alpha \int_0^{b} t \mathrm{d}F^{\dagger}(t)) = \rho \cdot \tilde{p}(b, F^{\dagger}).\end{aligned}$$

By taking expectation over the valuation, the expected revenue is scaled by ρ as well. $\square$

We finish this section by providing two simple monotonicity properties of the sample-bid mechanism and defer other more complicated characterizations required in our analysis to the later sections.

Lemma 4.5. *In the sample-bid mechanism, given any parameter α and distribution F, the expected payment for bid b is monotonically non-decreasing in b.*

Proof. By definition, the expected payment $\tilde{p}(b, F)$ of bid b over the randomness of the sample $s \sim F$ is

$$\tilde{p}(b, F) = \alpha b \cdot (1 - F(b)) + \alpha \int_{\underline{v}}^{\max\{b, \underline{v}\}} t \mathrm{d}F(t)$$

Taking the derivative with respect to bid b, we have

$$\frac{\partial \tilde{p}(b, F)}{\partial b} = \alpha(1 - F(b)) - \alpha b f(b) + \alpha b f(b) = \alpha(1 - F(b)) \geq 0.$$

which finishes the proof. $\square$

Lemma 4.6. *In the sample-bid mechanism, given any parameter α and distribution F, the optimal bid $b(v, F)$ is monotonically non-decreasing in value v.*

Proof. By Myerson (1981), the equilibrium allocation of the agent is non-decreasing in value v. Moreover, given the auction format, the equilibrium allocation of the agent is increasing in the bid, and thus the optimal bid $b(v, F)$ is non-decreasing in the value v. $\square$

4.3. Revenue of the Sample-bid Mechanism for MHR distributions

In this section, we analyze the prior-independent approximation ratio of the sample-bid mechanism over the class of MHR distributions.

Definition 4.6. *A distribution F is MHR if the hazard rate $\frac{f(v)}{1-F(v)}$ is monotone non-decreasing in v.*

Theorem 4.7. *For the sample-bid mechanism with $\alpha = 0.824$, the prior-independent approximation ratio over the class of MHR distributions is between [1.295, 1.296].*

The lower bound in Theorem 4.7 is shown in the following example.

Example 4.7. *For the sample-bid mechanism with $\alpha = 0.824$, let F be the valuation distribution such that $F(v) = 1 - e^{-v}$ for $v \in [0, 0.43)$ and $F(v) = 1$ for $v \in [0.43, \infty)$. It is easy to verify that F is MHR. Moreover, the optimal revenue is 0.2797 while the expected revenue of the sample-bid mechanism, which equals the expected revenue of posting a price equal to 0.824 fraction of the expected welfare, is 0.2159. Thus, the prior-independent approximation ratio of the sample-bid mechanism with $\alpha = 0.824$ is at least 1.295.*

Before the proof of the upper bound in Theorem 4.7, we first introduce a characterization of the agent's optimal bid when the sample distribution F is MHR; and a technical property for MHR distributions.

Lemma 4.8. *In the sample-bid mechanism, given any parameter α and MHR distribution F, the optimal bid $b(v, F)$ for the agent with value v is*

$$b(v, F) = \begin{cases} 0 & \text{if } v < \alpha \mathbf{E}_{s\sim F}[s]\,, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. By the proof of Lemma 4.3, the derivative of the utility given the bid b is

$$f(b)\left(v - \alpha \cdot \frac{1 - F(b)}{f(b)}\right),$$

where the sign of the above expression flips from negative to positive only once when the bid b increases from 0 to infinity since F is MHR. Thus the utility is a quasi-convex function of the bid, which implies that the maximum utility is attained at extreme points, i.e., bid 0 or ∞. Note that the utility for bidding 0 is always 0, while the utility for bidding ∞ is $u(v, \infty, F) = v - \alpha \mathbf{E}_{s\sim F}[s]$. Hence, the agent bid ∞ if and only her value v is at least $\alpha \mathbf{E}_{s\sim F}[s]$. □

Lemma 4.9 (Allouah and Besbes, 2019)**.** *For any MHR distribution with any pair of quantile and values $(v_1, q_1), (v_2, q_2)$ such that $q_1 = q(v_1) \leq q_2 = q(v_2)$ and $v_1 \geq v_2$. Then for any $v \geq v_2$, we have $q(v) \geq q_2 \cdot e^{\frac{v - v_2}{v_1 - v_2} \cdot \ln(\frac{q_1}{q_2})}$.*

Lemma 4.10. *The expected value for any MHR distribution with monopoly quantile $\hat{q}^\star$ is $w \geq \frac{\hat{q}^\star - 1}{\hat{q}^\star \cdot \ln \hat{q}^\star}$.*

Proof. The expected value of the agent is

$$\int_0^\infty q(v)\mathrm{d}v \geq \int_0^{\frac{1}{\hat{q}^\star}} e^{v\hat{q}^\star \cdot \ln \hat{q}^\star}\mathrm{d}v$$
$$= \frac{1}{\hat{q}^\star \cdot \ln \hat{q}^\star}(e^{\ln \hat{q}^\star} - e^0) = \frac{\hat{q}^\star - 1}{\hat{q}^\star \cdot \ln \hat{q}^\star},$$

where the inequality holds by applying Lemma 4.9 with $q_1 = \hat{q}^\star, v_1 = \frac{1}{\hat{q}^\star}$ and $q_2 = 1, v_2 = 0$. □

Now, we are ready to show Theorem 4.7.

Proof of the upper bound in Theorem 4.7. Fix any MHR distribution F. Let $w \triangleq \mathbf{E}_{v\sim F}[v]$. Note that by Lemma 4.8, our mechanism is equivalent to posting price αw to the agent. Next we analyze the approximation ratio by considering the cases $\alpha w \geq \hat{v}^\star$ and $\alpha w < \hat{v}^\star$ and optimize the parameter α such that the approximation ratio of both cases coincide. Recall that it is without loss of generality to normalize the expected revenue of the optimal mechanism to 1, i.e., $\hat{q}^\star \cdot \hat{v}^\star = 1$.

First we consider the case when $\alpha w < \hat{v}^\star = 1/\hat{q}^\star$. By Lemma 4.10, we have $w \geq \frac{\hat{q}^\star - 1}{\hat{q}^\star \cdot \ln \hat{q}^\star}$ and by combining Lemma 4.9 with $(v_1, q_1) = (\hat{v}^\star, \hat{q}^\star)$ and $(v_2, q_2) = (0, 1)$, we have $q(\alpha w) \geq e^{\alpha(\hat{q}^\star - 1)}$. Thus, the expected revenue in this case is

$$\alpha w \cdot q(\alpha w) \geq \frac{\alpha(\hat{q}^\star - 1)}{\hat{q}^\star \cdot \ln \hat{q}^\star} \cdot e^{\alpha(\hat{q}^\star - 1)}.$$

Then we consider the case when $\alpha w \geq \hat{v}^\star = 1/\hat{q}^\star$. In this case, combining Lemma 4.9 with $(v_1, q_1) = (w, q_w)$ and $(v_2, q_2) = (\hat{v}^\star, \hat{q}^\star)$, where $q_w \geq 1/e$ is the quantile of the welfare (see Barlow and Marshall, 1965), for any value $v \geq \hat{v}^\star$, we have $q(\alpha w) \geq \hat{q}^\star \cdot e^{\frac{\alpha w - \hat{v}^\star}{w - \hat{v}^\star} \cdot \ln(\frac{q_w}{\hat{q}^\star})}$.

Thus the expected revenue is

$$\alpha w \cdot q(\alpha w) \geq \alpha w \cdot \hat{q}^\star \cdot e^{\frac{\alpha w - \hat{v}^\star}{w - \hat{v}^\star} \cdot \ln(\frac{q_w}{\hat{q}^\star})} \geq \alpha w \cdot \hat{q}^\star \cdot e^{\frac{\alpha w - 1/\hat{q}^\star}{w - 1/\hat{q}^\star} \cdot \ln(\frac{1}{e \cdot \hat{q}^\star})}.$$

By setting $\alpha = 0.824$ and numerically evaluating the above expressions for all possible values of w and $\hat{q}^\star$ with respective to the given constraints, we have that the expected revenue in both cases are at least 0.7717, which guarantees approximation ratio 1.296. □

4.4. Revenue of the Sample-bid Mechanism for Regular distributions

In this section, we analyze the prior-independent approximation of the sample-bid mechanism over the class of regular distributions.

Theorem 4.11. *For the sample-bid mechanism with $\alpha = 0.7$, the prior-independent approximation ratio over the class of regular distributions is between* $[1.628, 1.835]$.

The lower bound in Theorem 4.11 is shown in the following example.

Example 4.8. *For the sample-bid mechanism with $\alpha = 0.7$, let F be the valuation distribution such that $F(v) = \frac{0.265}{v - 0.735}$ for $v \in [1, \infty)$. It is easy to verify that F is regular. Moreover, the optimal revenue is* 1 *while the expected revenue of the sample-bid mechanismis* 0.614*. Thus, the prior-independent approximation ratio of the sample-bid mechanism with $\alpha = 0.7$ is at least* 1.628*.*

In Section 4.4.1, we introduce some technical characterizations of the sample-bid mechanism which will be used in the subsequent analysis. In Sections 4.4.2 and 4.4.3, we study the prior-independent approximation ratio of the sample-bid mechanism over the class of regular distributions with monopoly quantile $\hat{q}^\star \geq 0.62$ and $\hat{q}^\star \leq 0.62$ respectively. By

Lemma 4.4, without loss of generality, we restrict our attention to the class of regular valuation distributions where the optimal revenue for the distributions is exactly 1 (i.e., $\hat{v}^\star \cdot \hat{q}^\star = 1$), and then lower-bound the expected revenue of the sample-bid mechanism with $\alpha = 0.7$.

Here we sketch the high-level approach to lower-bound the expected revenue of the sample-bid mechanism in both regimes (Sections 4.4.2 and 4.4.3). Given a regular distribution F, we define a value threshold $v^*(F)$ as the smallest value whose optimal bid is at least monopoly reserve $\hat{v}^\star(F)$, i.e.,

$$v^*(F) \triangleq \inf\{v : b(v, F) \geq \hat{v}^\star(F)\}$$

Denote $q(v^*(F), F)$ by $q^*(F)$. By Lemma 4.5 and Lemma 4.6, the expected revenue $\mathbf{Rev}_F(\text{SB})$ of the sample-bid mechanism SB for valuation F can be lower-bounded as follows,

$$\mathbf{Rev}_F(\text{SB}) = \int_0^1 p(v(q, F), F)\, dq \geq p(v^*(F), F) \cdot q^*(F) + \int_{q^*(F)}^1 p(v(q, F), F)\, dq.$$

where $p(v, F)$ is the expected payment of the agent, with value v and valuation distribution F, in the sample-bid mechanism. We then analyze $p(v^*(F), F)$, $q^*(F)$, and $p(v(q, F), F)$ for $q \geq q^*(F)$ by providing lower bounds as the functions of $\hat{q}^\star(F)$ and other some parameters of F.[4] Finally, by numerically evaluating the value of lower bounds for all possible possible parameters, we conclude that the expected revenue in the sample-bid mechanism for all regular distribution (with monopoly revenue 1) is at least 0.545, which implies

[4]Let R be the revenue curve induced by valuation distribution F. In Section 4.4.2, we lower-bound the expected revenue as a function of $\hat{q}^\star(F)$ and $R(0)$. In Section 4.4.2, we lower-bound the expected revenue as a function of $\hat{q}^\star(F)$, $q(\hat{v}^\star(F)/0.7, F)$ and $w \triangleq \int_{q(v/0.7, F)}^{\hat{q}^\star(F)} \frac{R(q)}{q}\, dq$.

the prior-independent approximation ratio $1/0.545 \approx 1.835$ of the sample-bid mechanism in Theorem 4.11. The details for discretizations and numerical evaluations can be found in Appendix B.1. Note that the bounds for the approximation ratio of the sample-based pricing mechanisms in Allouah and Besbes (2019) are also obtained by numerical analysis, which requires solving a relatively more complicated dynamic program. In contrast, our numerical analysis only requires brute force enumeration of a few parameters.

As we discussed in Section 4.1, every valuation distribution F can be represented by its induced revenue curve R where $R(q) \triangleq q\, F^{-1}(1-q)$ for all $q \in [0,1]$. In the remaining of the section, all statements, notations and analysis (except Lemma 4.13) will be presented in the language of revenue curves instead of valuation distributions.

4.4.1. Technical Properties of the Sample-bid Mechanism

In this subsection, we introduce some technical characterizations of the sample-bid mechanism which will be used in the later analysis.

To establish a lower bound on the expected revenue of of a truthful mechanism, a classic approach – revenue curve reduction – (e.g. Alaei, Hartline, Niazadeh, Pountourakis, and Yuan, 2018; Allouah and Besbes, 2018) is as follows: (i) start with an arbitrary revenue curve R_1, (ii) convert it to another revenue R_2 with closed-form formula while the optimal revenue remains the same, (iii) argue that the expected revenue for R_2 is at most the expected revenue for R_1 while the optimal revenue remains the same, and finally (iv) evaluate the expected revenue for R_2 for all possible parameters. In this section, we want to apply a similar approach to the sample-bid mechanism because it is a non-truthful mechanism. A new technical difficulty arises in step (iii). When comparing R_1 and R_2, for truthful mechanisms, it is sufficient to study the change in the expected payment (i.e. $\tilde{p}(b, R_1)$ and $\tilde{p}(b, R_2)$) for each bid b. However, for non-truthful mechanisms (e.g. sample-bid mechanism), the optimal bid of the agent changes when the revenue curve R_1 is replaced by R_2. In Lemma 4.12, we provide a characterization of optimal bid when we switch from R_1 to R_2 in a specific way (illustrated in Figure 4.1). We use it as a building block repeatedly in Section 4.4.2 and Section 4.4.3. Intuitively, the following lemma characterizes the phenomenon that increasing the revenue curve for high values does not affect the agent's preference for low bids.

Lemma 4.12. *In the sample-bid mechanism, consider any quantile $q^\dagger \in [0, 1]$ and any pair of revenue curves R_1, R_2 such that $R_1(q) \leq R_2(q)$ for any quantile $q \leq q^\dagger$ and $R_1(q^\dagger) = R_2(q^\dagger)$. Letting $b^\dagger = {R_1(q^\dagger)}/{q^\dagger}$. For any value v and any bid $b^\ddagger \geq b^\dagger$, if an agent with value v and revenue curve R_1 prefers bid $b^\dagger$ than $b^\ddagger$, i.e., $u(v, b^\dagger, R_1) \geq u(v, b^\ddagger, R_1)$, then an agent with value v and revenue curve R_2 also prefers bid $b^\dagger$ than $b^\ddagger$, i.e., $u(v, b^\dagger, R_2) \geq u(v, b^\ddagger, R_2)$.*

Proof. By the construction of our mechanism, the utility of an agent who has value v, revenue curve R and bids b is

$$u(v, b, R) = v \cdot (1 - q(b, R)) - \tilde{p}(b, R)$$

and

$$\tilde{p}(b, v) = \alpha b \cdot q(b, R) + \alpha \int_{q(b,R)}^{1} \frac{R(q)}{q} dq.$$

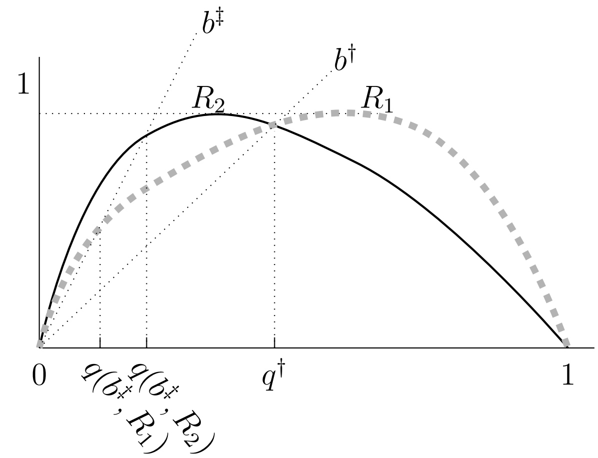


Figure 4.1. Graphical illustration for Lemma 4.12. The gray dashed thick (resp. black solid) curve is revenue curve R_1 (resp. R_2). The slopes of two dotted lines from $(0,0)$ are $b^\ddagger$ and $b^\dagger$ respectively.

By the assumption that $R_1(q) \leq R_2(q)$ for any quantile $q \leq q^\dagger$ and $b^\ddagger \geq b^\dagger$, we have $q(b^\ddagger, R_1) \leq q(b^\ddagger, R_2) \leq q^\dagger$. See Figure 4.1 for a graphical illustration. Thus,

$$\tilde{p}(b^\ddagger, R_1) - \tilde{p}(b^\dagger, R_1) = \alpha \cdot \left(-b^\dagger \cdot q^\dagger + \int_0^{q^\dagger} \min \left\{ \frac{R_1(q)}{q}, b^\ddagger \right\} \, dq \right)$$

$$\leq \alpha \cdot \left(-b^\dagger \cdot q^\dagger + \int_0^{q^\dagger} \min \left\{ \frac{R_2(q)}{q}, b^\ddagger \right\} \, dq \right) = \tilde{p}(b^\ddagger, R_2) - \tilde{p}(b^\dagger, R_2).$$

Thus,

$$u(b^\dagger, v, R_1) - u(b^\ddagger, v, R_1) = v \cdot (1 - q^\dagger) - \tilde{p}(b^\dagger, R_1) - v \cdot (1 - q(b^\ddagger, R_1)) + \tilde{p}(b^\ddagger, R_1)$$

$$\leq v \cdot (1 - q^\dagger) - \tilde{p}(b^\dagger, R_2) - v \cdot (1 - q(b^\ddagger, R_2)) + \tilde{p}(b^\ddagger, R_2) = u(b^\dagger, v, R_2) - u(b^\ddagger, v, R_2)$$

and hence $u(b^\dagger, v, R_1) \geq u(b^\ddagger, v, R_1)$ implies $u(b^\dagger, v, R_2) \geq u(b^\ddagger, v, R_2)$. $\square$

Lemma 4.13. *In the sample-bid mechanism with any parameter $\alpha \in [0,1]$, for an agent with concave revenue curve R and value v greater than the monopoly reserve $\hat{v}^\star$, she weakly prefers the bid ${}^{v}\!/\!{}_{\alpha}$ than any bid $b^\dagger \in [\hat{v}^\star, {}^{v}\!/\!{}_{\alpha}]$, i.e., $u(v, {}^{v}\!/\!{}_{\alpha}, R) \geq u(v, b^\dagger, R)$.*

Proof. Let F be a regular distribution. By the definition, the utility of the agent who has value v, valuation distribution F and bids b is

$$u(v, b, F) = v \cdot F(b) - \tilde{p}(b, F)$$

By considering the first order condition as in Lemma 4.3, we have

$$\frac{\partial u(v, b, F)}{\partial b} = f(b)\left(v - \alpha \cdot \frac{1 - F(b)}{f(b)}\right).$$

Thus, we can compute the difference between $u(v, {}^{v}\!/\!{}_{\alpha}, F)$ and $u(v, b, F)$ for any value $v \geq \hat{v}^\star$ and bid $b \in [\hat{v}^\star, {}^{v}\!/\!{}_{\alpha}]$ as follows,

$$\begin{aligned} u(v, {}^{v}\!/\!{}_{\alpha}, F) - u(v, b, F) &= \int_b^{\frac{v}{\alpha}} \alpha f(t)\left(\frac{v}{\alpha} - \frac{1 - F(t)}{f(t)}\right) dt \\ &\geq \int_b^{\frac{v}{\alpha}} \alpha f(t)\left(t - \frac{1 - F(t)}{f(t)}\right) dt \\ &\geq 0 \end{aligned}$$

where the last inequality uses the fact that $t - \frac{1-F(t)}{f(t)} \geq 0$ for all $t \geq \hat{v}^\star$ if F is regular. $\square$

4.4.2. Regular Distributions with Monopoly Quantile $\hat{q}^\star \geq 0.62$

In this subsection, we analyze the approximation ratio of the sample-bid mechanism over the class of regular distributions with monopoly quantile $\hat{q}^\star \geq 0.62$.

Lemma 4.14. *For the sample-bid mechanism with $\alpha = 0.7$, the approximation ratio over the class of regular distributions with monopoly quantile $\hat{q}^\star \geq 0.62$ is at most 1.835.*

Fix an arbitrary revenue curve R, let

$$v^*(R) \triangleq \inf\{v : b(v, R) \geq \hat{v}^\star(R)\}$$

be the smallest value whose optimal bid $b(v, R)$ for revenue curve R is at least the monopoly reserve $\hat{v}^\star(R)$. Since Lemma 4.6 guarantees that $b(v, R)$ is weakly non-decreasing in v, $v^*(R)$ is well-defined, $b(v, R) \geq \hat{v}^\star(R)$ for all $v \geq v^*(R)$, and $b(v, R) < \hat{v}^\star(R)$ for all $v < v^*(R)$. Denote $q(v^*(R), R)$ by $q^*(R)$. We decompose the proof of Lemma 4.14 by considering the following two subregimes – Lemma 4.15 for revenue curve R with $v^*(R) \leq \hat{v}^\star(R)$; and Lemma 4.17 for revenue curve R with $v^*(R) \geq \hat{v}^\star(R)$.

Lemma 4.15. *Given any concave revenue curve R such that $\hat{q}^\star(R) \geq 0.62$ and $v^*(R) \leq \hat{v}^\star(R)$, the revenue of the sample-bid mechanism with $\alpha = 0.7$ is a 1.835-approximation of the optimal revenue.*

Proof. Fix an arbitrary concave revenue curve R satisfying the requirement in the lemma statement, i.e., $\hat{q}^\star(R) \geq 0.62$ and $v^*(R) \leq \hat{v}^\star(R)$. Consider an arbitrary value $v \geq v^*(R)$. By Lemma 4.6, the optimal bid of an agent with value v is at least $\hat{v}^\star(R)$. Thus, together with Lemma 4.5, her expected payment in sample-bid mechanism is at least the expected payment $\tilde{p}(\hat{v}^\star(R), R)$ of bidding $\hat{v}^\star(R)$, and

$$\tilde{p}(\hat{v}^\star(R), R) = 0.7\hat{v}^\star(R)\hat{q}^\star(R) + 0.7\int_{\hat{q}^\star(R)}^{1} \frac{R(q)}{q}\,dq = 0.7 + 0.7\int_{\hat{q}^\star(R)}^{1} \frac{R(q)}{q}\,dq$$
$$\geq 0.7 + 0.7\int_{\hat{q}^\star(R)}^{1} \frac{\frac{1-q}{1-\hat{q}^\star(R)}}{q}\,dq = -\frac{0.7\log(\hat{q}^\star(R))}{1-\hat{q}^\star(R)}\ .$$

where the inequality uses the fact that (1) R is concave, which implies that $R(q) \geq \frac{1-q}{1-\hat{q}^\star(R)}$ for all $q \geq \hat{q}^\star(R)$; and (2) $\hat{v}^\star(R)\hat{q}^\star(R)$ is normalized to 1 for the revenue curve R. Since $v^*(R) \leq \hat{v}^\star(R)$, each value with quantile smaller than $\hat{q}^\star(R)$ has $\tilde{p}(\hat{v}^\star(R), R)$ as a lower bound of its payment in the sample-bid mechanism. Thus, a lower bound of the expected revenue $\mathbf{Rev}_R(\text{SB})$ for revenue curve R in the sample-bid mechanism is

$$\mathbf{Rev}_R(\text{SB}) = \int_0^1 p(v(q, F), F)\,dq \geq p(v^*(R), R)\cdot q^*(R)$$
$$\geq \tilde{p}(\hat{v}^\star(R), R)\cdot \hat{q}^\star(R) \geq -\frac{0.7\log(\hat{q}^\star(R))\hat{q}^\star(R)}{1-\hat{q}^\star(R)}$$

which is at least 0.545 for all $\hat{q}^\star(R) \geq 0.62$. This finishes the proof, since we (without loss of generality) consider revenue curve R with optimal revenue equal to 1, i.e., $\hat{v}^\star(R)\cdot\hat{q}^\star(R) = 1$. □

Before diving into the subregime where $v^*(R) \geq \hat{v}^\star(R)$, we provide a characterization (Lemma 4.16) of the optimal bid for concave revenue curves with monopoly quantile greater than 0.62. Specifically, Lemma 4.16 guarantees that $b(v, R) = 0$ for all value $v < v^*(R)$.

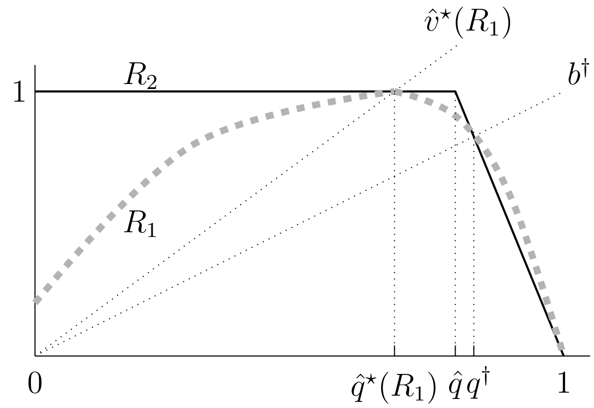


Figure 4.2. Graphical illustration for Lemma 4.16. The gray dashed (resp. black solid) curve is revenue curve R_1 (resp. R_2). The slopes of two dotted lines from (0, 0) are $\hat{v}^\star(R_1)$ and $b^\dagger$ respectively.

Lemma 4.16. *In the sample-bid mechanism with parameter* $\alpha = 0.7$*, given any value* v *and any concave revenue curve* R *with* $\hat{q}^\star(R) \geq 0.62$*, the optimal bid* $b(v, R)$ *for an agent with value* v *and revenue curve* R *satisfies* $b(v, R) \in \{0\} \cup [\hat{v}^\star(R), \infty)$*.*

Proof. We prove the lemma by contradiction. See Figure 4.2 for a graphical description of the following construction. Suppose there exists an agent who has value v, revenue curve R_1 s.t. $\hat{q}^\star(R_1) \geq 0.62$ and strictly prefers a bid of $b^\dagger \in (0, \hat{v}^\star(R_1))$ over all other bids. Denote $q(b^\dagger, R_1)$ by $q^\dagger$. Let $\hat{q} \triangleq 1 - \frac{1-q^\dagger}{R_1(q^\dagger)}$. Now consider another revenue curve R_2 defined as follows,

$$R_2(q) \triangleq \begin{cases} 1 & q \in [0, \hat{q}] \ , \\ \frac{1-q}{1-\hat{q}} & q \in [\hat{q}, 1] \ . \end{cases}$$

By construction, R_2 is a concave revenue curve s.t. (i) $\hat{q} \geq 0.62$; (ii) $b^\dagger \leq {}^1/_{\hat{q}}$; (iii) $R_1(q) \leq R_2(q)$ for all $q \in [0, q^\dagger]$; and (iv) $R_1(q) \geq R_2(q)$ for all $q \in [q^\dagger, 1]$.

Applying Lemma 4.12 on $R_1, R_2, q^\dagger, v$ and all $b^\ddagger \geq b^\dagger$, we conclude that the optimal bid for an agent with value v and revenue curve R_2 is in $[0, b^\dagger]$. Furthermore, note that $u(v, b^\dagger, R_2) \geq u(v, b^\dagger, R_1) > 0$ where the first inequality holds by the construction of R_2,[5] and the second inequality holds by our assumption that $b^\dagger$ is strictly preferred for R_1. Hence, there exists an optimal bid in $(0, b^\dagger]$ that is strictly preferred to biding zero and weakly preferred to all other bids for R_2. Next we argue that this leads to a contradiction by considering $v \leq 1/\hat{q}$ and $v \geq 1/\hat{q}$ separately.

Case (i) $v \leq 1/\hat{q}$*:* Note that for any bid $b \in [0, 1/\hat{q}]$, the utility $u(v, b, R_2)$ has a closed-form expression as follows,

$$u(v, b, R_2) = v\frac{b(1-\hat{q})}{b(1-\hat{q})+1} + 0.7\log\left(\frac{1}{b(1-\hat{q})+1}\right).$$

Considering the first order condition of $u(v, b, R_2)$ with respect to bid b, after basic simplification, we have

$$b = \frac{v}{0.7} - \frac{1}{1-\hat{q}}.$$

This leads to a contradiction since for all $\hat{q} \in [0.62, 1]$[6] and $v \in [0, 1/\hat{q}]$, we have $\frac{v}{0.7} - \frac{1}{1-\hat{q}} < 0$, i.e., bidding 0 is weakly preferred than any bid $b \in (0, b^\dagger)$.

Case (ii) $v \geq 1/\hat{q}$*:* Let $b^\ddagger \triangleq v/0.7$, and $q^\ddagger \triangleq q(b^\ddagger, R_2) = 0.7/v$. Since $v \geq 1/\hat{q}$, the construction of R_2 guarantees that $b^\ddagger \cdot q^\ddagger = 1 = R_2(q^\ddagger)$. Note that the utility $u(v, b^\ddagger, R_2)$ has a closed-form

[5] The allocation of bidding $b^\dagger$ is the same for both revenue curves, while the payment of bidding $b^\dagger$ is higher for revenue curve R_1.

[6] Note that $\hat{q}^\star \geq 0.62$ implies that $\hat{q} \geq 0.62$.

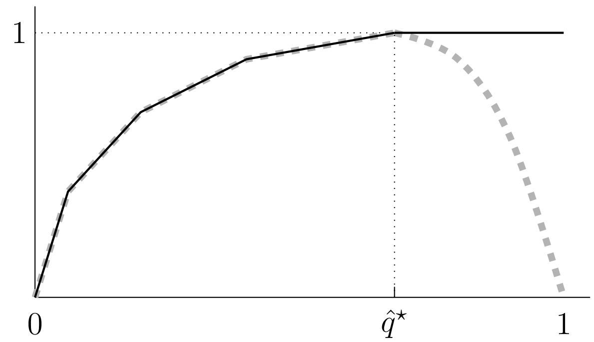


Figure 4.3. Graphical illustration for Lemma 4.17, Step 1. The gray dashed (resp. black solid) curve is the revenue curve R_1 (resp. R_2). By construction, $\hat{q}^\star(R_1) = \hat{q}^\star(R_2)$.

expression as follows,

$$u(v, b^\ddagger, R_2) = v - 2vq^\ddagger + 0.7(1-\hat{q}) + 0.7\log(\hat{q})$$
$$- 0.7(\hat{q} - q^\ddagger)(1 - b^\ddagger q^\ddagger)\left(\hat{q} - q^\ddagger - q^\ddagger \log(\hat{q}) + q^\ddagger \log(q^\ddagger)\right).$$

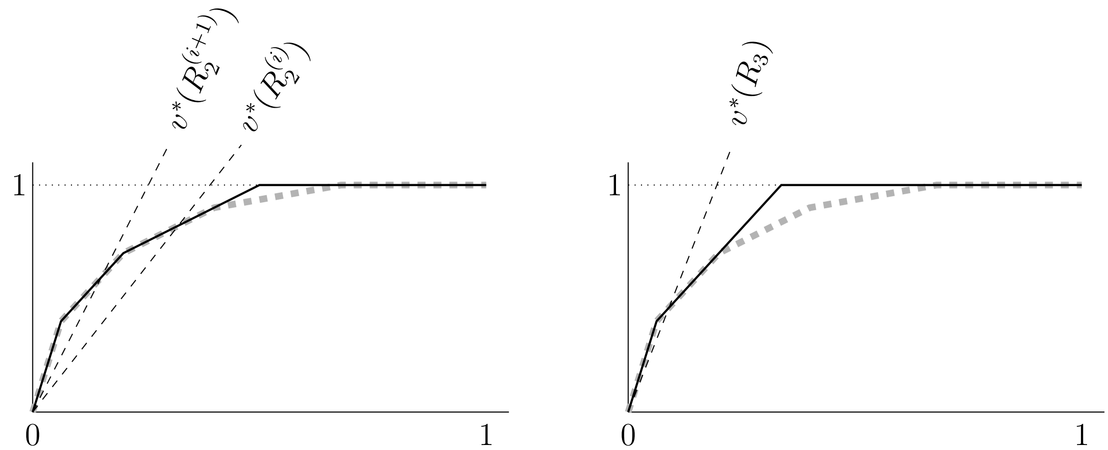
This leads to a contradiction since for all $\hat{q} \in [0.62, 1]$, $v \in [1/\hat{q}, \infty)$, and $\left(\frac{v}{0.7} - \frac{1}{1-\hat{q}}\right) \in [0, 1/\hat{q}]$, we have $u(v, b^\ddagger, R_2) \geq u\left(v, \frac{v}{0.7} - \frac{1}{1-\hat{q}}, R_2\right)$,[7] i.e., bidding 0 or $v/0.7$ is weakly preferred than any bid $b \in (0, b^\dagger)$. $\square$

Now, we provide the approximation guarantee for revenue curve R with $v^*(R) \geq \hat{v}^\star(R)$.

Lemma 4.17. *Given any concave revenue curve R such that $\hat{q}^\star(R) \geq 0.62$ and $v^*(R) \geq \hat{v}^\star(R)$, the revenue of the sample-bid mechanism with $\alpha = 0.7$ is a 1.835-approximation of the optimal revenue.*

Proof. The proof is done in four major steps:

[7]By first order condition, for revenue curve R_2, if $\left(\frac{v}{0.7} - \frac{1}{1-\hat{q}}\right) > 1/\hat{q}$, then bidding $b^\ddagger$ already achieves higher utility for the agent compared to bidding below $b^\ddagger$. Thus it is sufficient to compare $b^\ddagger$ with $\left(\frac{v}{0.7} - \frac{1}{1-\hat{q}}\right)$ in the case that the latter is in $[0, 1/\hat{q}]$.



(a) The gray dashed (resp. black solid) curve is revenue curve $R_2^{(i)}$ (resp. $R_2^{(i+1)}$). The slopes of two dotted lines from (0, 0) are $v^*(R_2^{(i+1)})$ and $v^*(R_2^{(i)})$ respectively. By construction, $v^*(R_2^{(i+1)}) \geq v^*(R_2^{(i)})$.

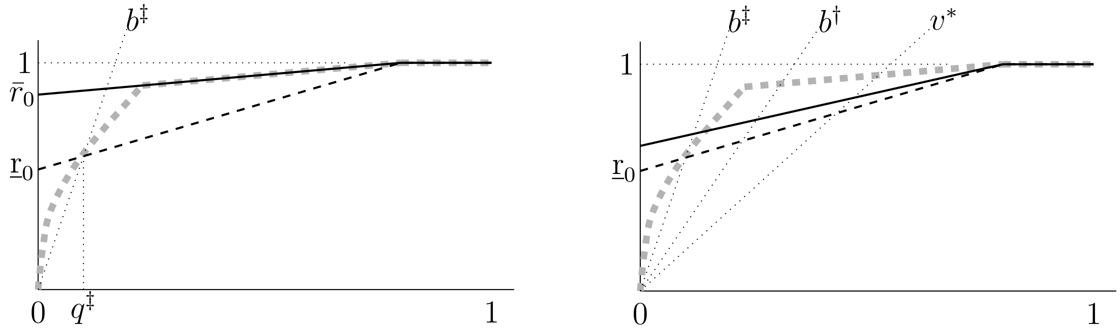
(b) The gray dashed (resp. black solid) curve is revenue curve R_2 (resp. R_3). The slope of the dotted line from (0, 0) is $v^*(R_3)$.

Figure 4.4. Graphical illustration for Lemma 4.17, Step 2.

Step 1- flattening the revenue curve for all quantile $q \geq \hat{q}^\star(R_1)$. Fix an arbitrary revenue curve R_1 satisfying the requirements in the lemma statement, i.e., $\hat{q}^\star(R) \geq 0.62$ and $v^*(R) \geq \hat{v}^\star(R)$. Consider another revenue curve R_2 defined as follows (see Figure 4.3 for a graphical illustration)

$$R_2(q) \triangleq \begin{cases} R_1(q) & q \in [0, \hat{q}^\star(R_1)] \ , \\ 1 & q \in [\hat{q}^\star(R_1), 1] \ . \end{cases}$$

We claim that the expected revenue of the sample-bid mechanism with $\alpha = 0.7$ for revenue curve R_2 is at most that of revenue curve R_1. To see this, consider the virtual surplus for both revenue curves. By our assumption that $v^*(R_1) \geq \hat{v}^\star(R_1)$, every quantile $q > \hat{q}^\star(R_1)$ has negative virtual value $R_1'(q)$ in R_1, bids zero (Lemma 4.16) and gains zero virtual



(a) The gray dashed (resp. black solid, black dashed) curve is revenue curve R_3 (resp. $R_4^{(\bar{r}_0)}$, $R_4^{(\underline{r}_0)}$). The slope of the dotted line from (0, 0) is $b^{\ddagger}$, i.e., the optimal bid for an agent with value $v^*(R_3)$ and revenue curve R_3. By construction, $v^*(R_4^{(\underline{r}_0)}) \leq v^*(R_3)) \leq v^*(R_4^{(\bar{r}_0)})$.

(b) The gray dashed (resp. black solid, black dashed) curve is revenue curve R_3 (resp. R_4, $R_4^{(\underline{r}_0)}$). By construction, $v^*(R_3) = v^*(R_4)$ ($\triangleq v^*$). The slope of three dotted lines from (0, 0) are $b^{\ddagger}$, $b^{\dagger}$ and v^*, where $b^{\ddagger}$ (resp. $b^{\dagger}$) is the optimal bid for an agent with value v^* and revenue curve R_3 (resp. R_4). By Lemma 4.3, $b^{\dagger} \leq b^{\ddagger}$.

Figure 4.5. Graphical illustration for Lemma 4.17, Step 3.

surplus while their virtual value $R_2'(q)$ becomes zero in R_2 and thus gains zero virtual surplus as well. On the other side, every quantile $q \leq \hat{q}^{\star}(R_1)$ has identical virtual value by construction. We claim that the allocation for each of these quantiles weakly decreases. To see this, note that the allocation of bidding any bid $b \geq \hat{v}^{\star}(R_1) = \hat{v}^{\star}(R_2)$ is the same for both revenue curves R_1 and R_2, and the expected payment increases by a constant when the revenue curve R_1 is replace by R_2. Thus the agent's preference among all bids $b \geq \hat{v}^{\star}(R_1)$ is the same in both revenue curves R_1 and R_2. However, the utility of bidding $b \geq \hat{v}^{\star}(R_2)$ is lower when the revenue curve is R_2, which implies that there may exist value v such that the agent may prefer bidding 0 to bidding above the monopoly reserve in R_2, while strictly prefer bidding above the monopoly reserve in R_1. By Lemma 4.16, the optimal bid for any value v is not in $(0, \hat{v}^{\star}(R_2))$. Thus, we conclude that $q^*(R_2) \leq q^*(R_1)$

and (1) the optimal bid (as well as the allocation) for every quantile $q \leq q^*(R_2)$ in both R_1 and R_2 remains the same; and (2) for every quantile $q \in [q^*(R_2), q^*(R_1))$, the optimal bid quantile q is 0 when the revenue curve is R_2. This guarantees that the virtual surplus for every quantile $q \leq \hat{q}^\star(R_1)$ weakly decreases since the virtual value is non-negative while the allocation decreases. Note that in sample-bid mechanism, the payment for lowest type is always 0, i.e., $p(0) = 0$. By Lemma 2.2, the expected revenue (a.k.a. virtual surplus) for R_2 is at most the expected revenue (a.k.a. virtual surplus) for R_1.

Step 2- flattening the revenue curve for all quantiles $q \geq q^*$. In this step, we start with revenue curve R_2 constructed in step 1, and consider a sequence of revenue curves $R_2^{(0)}, R_2^{(1)}, \ldots$ where $R_2^{(0)} \triangleq R_2$ and $R_2^{(i+1)}$ is recursively defined on $R_2^{(i)}$ as follows,

$$R_2^{(i+1)}(q) \triangleq \begin{cases} R_2^{(i)}(q) & q \in \left[0, q^*(R_2^{(i)})\right] \ , \\ R_2^{(i)\prime}(q^*(R_2^{(i)})) \cdot (q - q^*(R_2^{(i)})) + R_2^{(i)}(q^*(R_2^{(i)})) & \\ & q \in \left[q^*(R_2^{(i)}), \frac{1-R_2^{(i)}(q^*(R_2^{(i)}))}{R_2^{(i)\prime}(q^*(R_2^{(i)}))} + q^*(R_2^{(i)})\right] \ , \\ 1 & q \in \left[\frac{1-R_2^{(i)}(q^*(R_2^{(i)}))}{R_2^{(i)\prime}(q^*(R_2^{(i)}))} + q^*(R_2^{(i)}), 1\right] \ . \end{cases}$$

where $R_2^{(i)\prime}(q^*(R_2^{(i)}))$ is the right-hand derivative of $R_2^{(i)}(q)$ at $q = q^*(R_2^{(i)})$. See Figure 4.4a for a graphical illustration. Invoking Lemma 4.13 and Lemma 4.16, with the same argument for values with positive virtual values in step 1, we can conclude that $q^*(R_2^{(i)})$ and the expected revenue for $R_2^{(i)}$ in the sample-bid mechanism is weakly decreasing in i.

Note that by construction, the sequence of revenue curves $R_2^{(0)}, R_2^{(1)}, \ldots$ converges to a revenue curve R_3 whose expected revenue in the sample-bid mechanism is at most the revenue for R_2, and satisfying the following characterization,

$$R_3(q) \triangleq \begin{cases} R_2(q) & q \in [0, q^*(R_3)] \ , \\ R_2'(q^*(R_3)) \cdot (q - q^*(R_3)) + R_2(q^*(R_3)) & q \in \left[q^*(R_3), \frac{1-R_2(q^*(R_3))}{R_2'(q^*(R_3))} + q^*(R_3)\right] \ , \\ 1 & q \in \left[\frac{1-R_2(q^*(R_3))}{R_2'(q^*(R_3))} + q^*(R_3), 1\right] \ . \end{cases}$$

See Figure 4.4b for a graphical illustration.

Step 3- flattening the revenue curve for all quantile $q \leq \hat{q}^\star(R_3)$. For any revenue curve R, let $p(v^*(R), R)$ be the expected payment in the sample-bid mechanism of an agent with value $v^*(R)$ and revenue curve R. Due to Lemma 4.5 and Lemma 4.6, $p(v^*(R), R) \cdot q^*(R)$ is a valid lower bound of the expected revenue in the sample-bid mechanism for an agent with revenue curve R. In this step, instead of analyzing the expected revenue, we argue that we can convert any revenue curve R_3 (constructed in step 2) into another revenue curve R_4, such that (i) $v^*(R_4) = v^*(R_3)$ ($\triangleq v^*$); (ii) $q^*(R_4) \leq q^*(R_3)$; and (iii) $p(v^*, R_4) \leq p(v^*, R_3)$. Finally, by showing that $p(v^*(R_4), R_4) \cdot q^*(R_4) \geq 0.545$, we finish the proof of the lemma.

Given the revenue curve R_3 constructed in step 2, for any $r_0 \in [0, 1]$, we define a revenue curve $R_4^{(r_0)}$ as follows,

$$R_4^{(r_0)} \triangleq \begin{cases} r_0 + (1 - r_0)\frac{q}{\hat{q}^\star(R_3)} & q \in [0, \hat{q}^\star(R_3)] \ , \\ 1 & q \in [\hat{q}^\star(R_3), 1] \ . \end{cases}$$

See the black curves in Figure 4.5a as an example. We claim that there exists $r_0^* \in [0, 1]$ s.t. $R_4^{(r0)}$ ($\triangleq R_4$) satisfies properties (i) (ii) (iii) mentioned above. To see this, consider the argument as follows.

By construction, for all every value v, every bid b, the utility $u(v, b, R_4^{(r_0)})$ is decreasing continuously in r_0. Thus, $v^*(R_4^{(r_0)})$ is decreasing continuously in r_0. Let $b^\ddagger$ be the optimal bid of an agent with value $v^*(R_3)$ and revenue curve R_3. Denote $q(b^\ddagger, R_3)$ by $q^\ddagger$. Consider revenue curve $R_4^{(\underline{r}_0)}$ where $\underline{r}_0 \triangleq 1 - \frac{\hat{q}^*(R_3)}{\hat{q}^*(R_3) - q^\ddagger}(1 - R_3(q^\ddagger))$. By construction, $R_4^{(\underline{r}_0)}(q) \geq R_3(q)$ for all $q \leq q^\ddagger$, and $R_4^{(\underline{r}_0)}(q) \leq R_3(q)$ for all $q \geq q^\ddagger$. See Figure 4.5a for a graphical illustration. Note that by construction,

$$
\begin{aligned}
u(v^*(R_3), b^\ddagger, R_4^{(\underline{r}_0)}) &= v^*(R_3) \cdot (1 - q^\ddagger) - \alpha b^\ddagger \cdot q^\ddagger - \alpha \int_{q^\ddagger}^1 \frac{R_4^{(\underline{r}_0)}(q)}{q} \, dq \\
&\geq v^*(R_3) \cdot (1 - q^\ddagger) - \alpha b^\ddagger \cdot q^\ddagger - \alpha \int_{q^\ddagger}^1 \frac{R_3(q)}{q} \, dq = u(v^*(R_3), b^\ddagger, R_3) = 0
\end{aligned}
$$

Thus, $v^*(R_4^{(\underline{r}_0)}) \leq v^*(R_3)$. Next, consider revenue curve $R_4^{(\bar{r}_0)}$ where $\bar{r}_0 \triangleq 1 - \frac{\hat{q}^*(R_3)}{\hat{q}^*(R_3) - q^*(R_3)}(1 - R_3(q^*(R_3))$. By construction, $R_4^{(\bar{r}_0)}(q) \geq R_3(q)$ for all $q \in [0, 1]$. See Figure 4.5a for a graphical illustration. Thus, $v^*(R_4^{(\bar{r}_0)}) \geq v^*(R_3)$ with the similar argument for $R_4^{(\underline{r}_0)}$ Therefore, we know that there exists $r_0^* \in [\underline{r}_0, \bar{r}_0]$ such that $v^*(R_4^{(r_0^*)}) = v^*(R_3)$. We denote $R_4^{(r_0^*)}$ by R_4 and show that R_4 satisfies properties (ii) $q^*(R_4) \leq q^*(R_3)$ and (iii) $p(v^*, R_4) \leq p(v^*, R_3)$ with the argument below.

Lemma 4.13 implies that $b^\ddagger > v^*(R_3)$. Combining with the fact that $r_0^* \geq \underline{r}_0$, we know that property (ii) $q^*(R_4) \leq q^*(R_3)$ is satisfied. See Figure 4.5b for a graphical illustration.

Combining the first order condition in Lemma 4.3 and construction of R_4, it is guaranteed that the optimal bid $b^\dagger$ of value v^* for revenue curve R_4 is at most $b^\ddagger$. Furthermore, $q(b^\dagger, R_4) \geq q(b^\ddagger, R_4) \geq q(b^\ddagger, R_3) = q^\ddagger$ by construction. By the definition, the optimal utility of value $v^*(R)$ for any revenue curve R is zero. Thus, $p(v^*, R_3) = v^* \cdot (1 - q^\ddagger) \geq v^* \cdot (1 - q(b^\dagger, R_4)) = p(v^*, R_4)$.

Step 4- lower-bounding the expected revenue on R_4. So far, we have shown that for an arbitrary revenue curve satisfying the assumptions in lemma statement, its expected revenue in the sample-bid mechanism is lower-bounded by $p(v^*(R_4), R_4) \cdot q^*(R_4)$ for R_4 pinned down by some $(r_0, \hat{q}^\star)$ as follows,

$$R_4 \triangleq \begin{cases} r_0 + (1 - r_0)\frac{q}{\hat{q}^\star} & q \in [0, \hat{q}^\star] \ , \\ 1 & q \in [\hat{q}^\star, 1] \ . \end{cases}$$

By numerically verifying $p(v^*(R_4), R_4) \cdot q^*(R_4) \geq 0.545$ for all $(r_0, \hat{q}^\star) \in [0, 1]^2$, we finish the proof. The details of this numerical evaluation is elaborated on in Appendix B.1. $\square$

4.4.3. Regular Distributions with Monopoly Quantile $\hat{q}^\star \leq 0.62$

In this subsection, we analyze the prior-independent approximation ratio of the sample-bid mechanism over the class of regular distributions with monopoly quantile $\hat{q}^\star \leq 0.62$.

Lemma 4.18. *For the sample-bid mechanism with $\alpha = 0.7$, the prior-independent approximation ratio over the class of regular distributions with monopoly quantile $\hat{q}^\star \leq 0.62$ is at most* 1.835.

Fix an arbitrary revenue curve R, let

$$v^*(R) \triangleq \inf\{v : b(v, R) \geq \hat{v}^\star(R)\}$$

be the smallest value whose optimal bid $b(v, R)$ for revenue curve R is at least $\hat{v}^\star(R)$. Since Lemma 4.6 guarantees that $b(v, R)$ is weakly non-decreasing in v, $v^*(R)$ is well-defined, $b(v, R) \geq \hat{v}^\star(R)$ for all $v \geq v^*(R)$. Furthermore, by Lemma 4.13, we know that $b(v, R) \geq {\hat{v}^\star(R)}/{0.7}$ for all $v \geq \max\{v^*(R), \hat{v}^\star(R)\}$. Denote $q(v^*(R), R)$ by $q^*(R)$. By

Lemma 4.5 and Lemma 4.6, the expected revenue $\mathbf{Rev}_R(\mathrm{SB})$ of the sample-bid mechanism for revenue curve R can be lower-bounded as follows,

$$\begin{aligned}
\mathbf{Rev}_R(\mathrm{SB}) &= \int_0^1 p(v(q,R),R)\,dq \\
&= \int_0^{\min\{q^*(R),\hat{q}^\star(R)\}} p(v(q,R),R)\,dq + \int_{\min\{q^*(R),\hat{q}^\star(R)\}}^{q^*(R)} p(v(q,R),R)\,dq \\
&\qquad + \int_{q^*(R)}^1 p(v(q,R),R)\,dq \\
&\geq \tilde{p}({\hat{v}^\star(R)}/{0.7},R)\cdot \min\{q^*(R),\hat{q}^\star(R)\} + \tilde{p}(\hat{v}^\star(R),R)\cdot \max\{0, q^*(R)-\hat{q}^\star(R)\} \\
&\qquad + \int_{q^*(R)}^1 p(v(q,R),R)\,dq.
\end{aligned}$$

Denote $q({\hat{v}^\star(R)}/{0.7},R)$ by $q^\ddagger(R)$, and $\int_{q^\ddagger(R)}^{\hat{q}^\star(R)} \frac{R(q)}{q}\,dq$ by $w(R)$. In Lemma 4.19, we lower-bound the expected payment $\tilde{p}({\hat{v}^\star(R)}/{0.7},R)$ and $\tilde{p}(\hat{v}^\star(R),R)$ as the function of $\hat{q}^\star(R)$, $q^\ddagger(R)$, $w(R)$ and $v^*(R)$. In Lemma 4.20, we lower-bound $q^*(R)$ as the function of $\hat{q}^\star(R)$, $q^\ddagger(R)$ and $v^*(R)$. In Lemma 4.21, we upper-bound of $v^*(R)$ as the function of $\hat{q}^\star(R)$, $q^\ddagger(R)$ and $w(R)$. In Lemma 4.22, we lower-bound $p(v(q,R),R)$ as a function of $\hat{q}^\star(R)$ for all quantile $q \in [\hat{q}^\star(R),1]$. Putting all pieces together, we show Lemma 4.18 by providing a lower bound of expected revenue in the sample-bid mechanism as a function of $\hat{q}^\star(R)$, $q^\ddagger(R)$ and $w(R)$, and numerically evaluating its value for all possible parameters. The details of the numerical evaluations in this section are similar to those of Lemma 4.17, which are elaborated on in Appendix B.1.

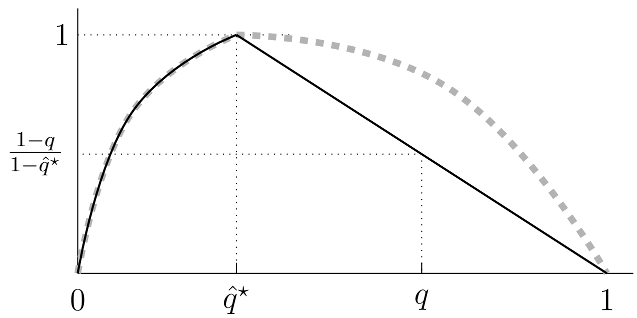


Figure 4.6. Graphical illustration for Lemma 4.19. The gray dashed (resp. black solid) curve is revenue curve R (resp. lower bound of R).

Lemma 4.19. *For the sample-bid mechanism with $\alpha = 0.7$, given any concave revenue curve R, the expected payment $\tilde{p}(b, R)$ of bidding $b \in [0, \hat{v}^\star(R)]$ is at least*

$$\tilde{p}(b,R) \geq \frac{0.7\log(b\cdot(1-\hat{q}^\star(R))+1)}{1-\hat{q}^\star(R)};$$

and the expected payment $\tilde{p}(\hat{v}^\star(R)/0.7, R)$ of bidding $\hat{v}^\star(R)/0.7$ is at least

$$\tilde{p}(\hat{v}^\star(R)/0.7, R) \geq \left(\frac{q^\ddagger(R)}{\hat{q}^\star(R)} + 0.7w(R) - \frac{0.7\log(\hat{q}^\star(R)))}{1-\hat{q}^\star(R)}\right).$$

Proof. By definition, for any $b \in [0, \hat{v}^\star(R)]$,

$$\begin{aligned}
\tilde{p}(b,R) &= 0.7b\cdot q(b,R) + 0.7\int_{q(b,R)}^{1} \frac{R(q)}{q}\,dq \\
&\geq 0.7b\cdot q(b,R) + 0.7\int_{q(b,R)}^{1} \frac{\frac{1-q}{1-\hat{q}^\star(R)}}{q}\,dq \\
&= 0.7b\cdot q(b,R) - 0.7\frac{1-q(b,R)}{1-\hat{q}^\star(R)} - \frac{0.7\log(q(b,R))}{1-\hat{q}^\star(R)} \\
&\geq \frac{0.7\log(b\cdot(1-\hat{q}^\star(R))+1)}{1-\hat{q}^\star(R)}
\end{aligned}$$

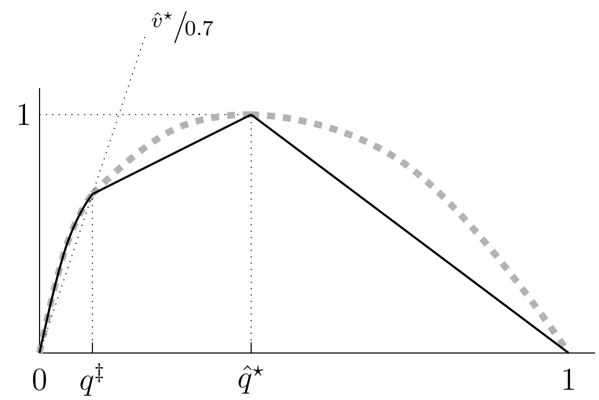


Figure 4.7. Graphical illustration for Lemma 4.20. The gray dashed (resp. black solid) curve is revenue curve R_1 (resp. R_2). The slope of the dotted line from (0, 0) is $\hat{v}^\star(R_1)/0.7$.

where the first inequality uses the fact that $R(q) \geq \frac{1-q}{1-\hat{q}^\star(R)}$ for all $q \geq \hat{q}^\star(R)$ from the regularity of R, and the second inequality use the fact that $b \cdot q(b, R) \geq \frac{1-q(b,R)}{1-\hat{q}^\star(R)}$, and $q(b, R) \leq (b \cdot (1 - \hat{q}^\star(R)) + 1)^{-1}$ from the regularity of R. See Figure 4.6 for a graphical illustration.

Similarly,

$$
\begin{aligned}
\tilde{p}(\hat{v}^\star(R)/0.7, R) &= 0.7\frac{\hat{v}^\star(R)}{0.7}q^\ddagger(R) + 0.7\int_{q^\ddagger(R)}^{1} \frac{R(q)}{q}\,dq \\
&= \frac{q^\ddagger(R)}{\hat{q}^\star(R)} + 0.7w(R) + 0.7\int_{\hat{q}^\star(R)}^{1} \frac{R(q)}{q}\,dq \\
&\geq \frac{q^\ddagger(R)}{\hat{q}^\star(R)} + 0.7w(R) - \frac{0.7\log(\hat{q}^\star(R))}{1-\hat{q}^\star(R)} \;. \qquad \square
\end{aligned}
$$

Lemma 4.20. *For any concave revenue curve R, the quantile $q(v, R)$ for value $v \leq \hat{v}^\star(R)$ is at least*

$$q(v, R) \geq \frac{1}{1 + v \cdot (1 - \hat{q}^\star(R))};$$

and the quantile $q(v, R)$ for value $v \in [\hat{v}^\star(R), \hat{v}^\star(R)/0.7]$ is at least

$$q(v, R) \geq \frac{2\hat{q}^\star(R) - q^\ddagger(R) \cdot (1 + 1/0.7)}{1 + v \cdot (1 - \hat{q}^\star(R))}.$$

Proof. Given any concave revenue curve R_1, consider another revenue curve R_2 defined as follows,

$$R_2(q) \triangleq \begin{cases} R_1(q) & q \in [0, q^\ddagger(R_1)] \ , \\ R_1(q^\ddagger(R_1)) + \frac{q - q^\ddagger(R_1)}{\hat{q}^\star(R_1) - q^\ddagger(R_1)}(1 - R_1(q^\ddagger(R_1))) & q \in [q^\ddagger(R_1), \hat{q}^\star(R_1)] \ , \\ \frac{1-q}{1-\hat{q}^\star(R_1)} & q \in [\hat{q}^\star(R_1), 1] \ . \end{cases}$$

Since R_1 is regular, we have $R_2(q) \leq R_1(q)$ for all $q \in [0, 1]$ by construction. See Figure 4.7 for graphical illustration. Thus, for any value $v \leq \hat{v}^\star R_1$, we have

$$q(v, R_1) \geq q(v, R_2) = \frac{1}{1 + v \cdot (1 - \hat{q}^\star(R_1))}.$$

Moreover, for any value $v \in [\hat{v}^\star(R_1), \hat{v}^\star(R_1)/0.7]$, we have

$$q(v, R_1) \geq q(v, R_2) = \frac{2\hat{q}^\star(R_1) - q_1(R_1) \cdot (1 + 1/0.7)}{1 + v \cdot (1 - \hat{q}^\star(R_1))}.$$ □

Lemma 4.21. *In the sample-bid mechanism with parameter $\alpha = 0.7$, given any value v and any concave revenue curve R, the optimal bid $b(v, R)$ for an agent with value v and*

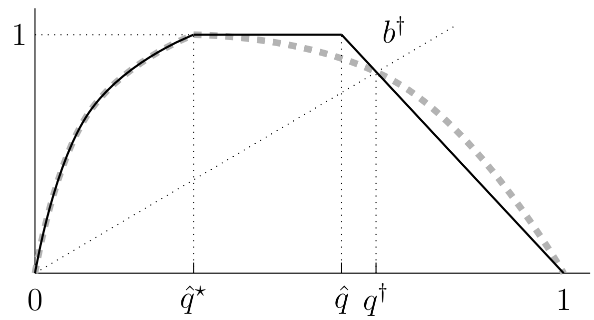


Figure 4.8. Graphical illustration for Lemma 4.21. The gray dashed (resp. black solid) curve is revenue curve R_1 (resp. R_2). The slope of the dotted line from (0, 0) is $b^\dagger$.

revenue curve R is at least $\hat{v}^\star(R)$ if for all $\hat{q} \in [\hat{q}^\star(R), 1]$,

(4.2)
$$v \cdot (1 - q^\ddagger(R)) - \hat{v}^\star(R) \cdot q^\ddagger(R) - 0.7 \left(w(R) + \log\left(\frac{\hat{q}}{\hat{q}^\star(R)}\right) - \frac{\ln(\hat{q})}{1-\hat{q}} \right) \geq v(1-\tilde{q}) + \frac{0.7 \log(\tilde{q})}{1-\hat{q}}$$

where $\tilde{q} \triangleq \left(1 + \min\{1/\hat{q}, \max\{0, \frac{v}{0.7} - \frac{1}{1-\hat{q}}\}\} \cdot (1-\hat{q})\right)^{-1}$.

Proof. Fix an arbitrary concave revenue curve R. We show that inequality (4.2) in the lemma statement is a sufficient condition that bidding $\hat{v}^\star(R)/0.7$ is weakly preferred than bidding any bids in $[0, \hat{v}^\star(R)]$. The argument is similar to Lemma 4.16.

We prove by contradiction, suppose there exists an revenue curve R_1, and value v such that inequality (4.2) in the lemma statement is satisfied but the optimal bid of an agent with value v and revenue curve R_1 is $b^\dagger \in [0, \hat{v}^\star(R_1))$. Denote $q(b^\dagger, R_1)$ by $q^\dagger$. Let $\hat{q} \triangleq 1 - \frac{1-q^\dagger}{R_1(q^\dagger)}$. By construction, $\hat{q} \geq \hat{q}^\star(R_1)$. Now consider another revenue curve R_2

defined as follows,

$$R_2(q) \triangleq \begin{cases} R_1(q) & q \in [0, \hat{q}^\star(R_1)] \ , \\ 1 & q \in [\hat{q}^\star(R_1), \hat{q}] \ , \\ \frac{1-q}{1-\hat{q}} & q \in [\hat{q}, 1] \ . \end{cases}$$

By construction, R_2 is a concave revenue curve s.t. (i) $R_1(q) = R_2(q)$ for all $q \in [0, \hat{q}^\star(R_1)]$; (ii) $R_1(q) \le R_2(q)$ for all $q \in [\hat{q}^\star(R_1), q^\dagger]$; and (iii) $R_1(q) \ge R_2(q)$ for all $q \in [q^\dagger, 1]$; See Figure 4.8 for a graphical illustration.

Applying Lemma 4.12 on $R_1, R_2, q^\dagger, v$ and all $b^\ddagger \ge b^\dagger$, we conclude that the optimal bid for an agent with value v and revenue curve R_2 is in $[0, b^\dagger]$.

Note that for any bid $b \in [0, 1/\hat{q}]$, the utility $u(v, b, R_2)$ has a closed-form expression as follows,

$$u(v, b, R_2) = v \cdot \frac{b(1-\hat{q})}{b(1-\hat{q})+1} + 0.7 \log\left(\frac{1}{b(1-\hat{q})+1}\right).$$

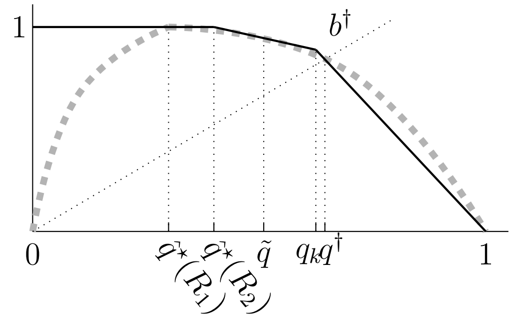
Considering the first order condition of $u(v, b, R_2)$ with respect to bid b, after basic simplification, we have

$$b = \frac{v}{0.7} - \frac{1}{1-\hat{q}}.$$

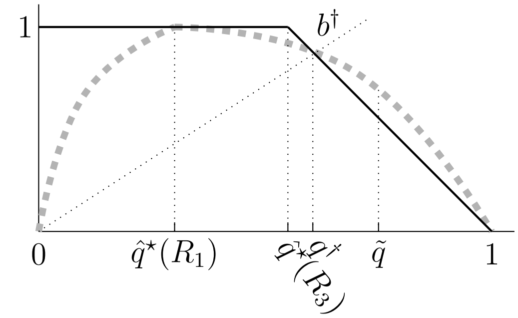
Thus, the optimal bid in $[0, 1/\hat{q}]$ for revenue curve R_2 is $\tilde{b} \triangleq \min\{1/\hat{q}, \max\{0, \frac{v}{0.7} - \frac{1}{1-\hat{q}}\}\}$. Plugging $u(v, b, R_2)$ with $b = \tilde{b}$, we get

$$v(1-\tilde{q}) + \frac{0.7\log(\tilde{q})}{1-\hat{q}},$$

i.e., the right hand side of inequality (4.2).



(a) Case (i) $b^\dagger \le \tilde{v}$.



(b) Case (ii) $b^\dagger \ge \tilde{v}$.

Figure 4.9. Graphical illustration for Lemma 4.22. The gray dashed (resp. black solid) curve is revenue curve R_1 (resp. R_2 in (a) and R_3 in (b)). The slope of the dotted line from (0, 0) is $b^\dagger$.

Moreover, note that the utility $u(v, \hat{v}^\star(R_1)/0.7, R_2)$ has a closed-form expression as follows,

$$v \cdot (1 - q^\ddagger(R)) - \hat{v}^\star(R) \cdot q^\ddagger(R) - 0.7 \left(w(R) + \log\left(\frac{\hat{q}}{\hat{q}^\star(R)}\right) - \frac{\ln(\hat{q})}{1-\hat{q}} \right)$$

i.e., the left hand side of inequality (4.2). This leads to a contradiction, which finishes the proof. ☐

Definition 4.9. *A* pentagon revenue curve R *parameterized by the quantile* $q_k \in [\hat{q}^\star(R), 1]$ *of kink and the revenue* $r_k \in \left[\frac{1-q_k}{1-\hat{q}^\star(R)}, 1\right]$ *on this kink is defined as follows*

$$R(q) \triangleq \begin{cases} 1 & q \in [0, \hat{q}^\star(R)]\ , \\ r_k + \frac{q - \hat{q}^\star(R)}{q_k - \hat{q}^\star(R)}(1 - r_k) & q \in [\hat{q}^\star(R), q_k]\ , \\ \frac{1-q}{1-q_k} \cdot r_k & q \in [q_k, 1]\ . \end{cases}$$

An example of a pentagon revenue curve is illustrated as the solid curve in Figure 4.9a as the solid line.

Lemma 4.22. *In the sample-bid mechanism, given any quantile $\hat{q} \in [0,1]$, quantile $\tilde{q} \in [\hat{q}, 1]$, and bid $b \in [0, 1/\hat{q}]$, if for all pentagon revenue curves R_P with $\hat{q}^\star(R_P) \geq \hat{q}$, the optimal bid of an agent with value $v(\tilde{q}, R_P)$ and revenue curve R_P is at least b; then for all concave revenue curves R with $\hat{q}^\star(R) = \hat{q}$, the optimal bid of an agent with value $v(\tilde{q}, R)$ and revenue curve R is at least b as well.*

Proof. Fix arbitrary $\hat{q} \in [0,1]$, $\tilde{q} \in [\hat{q}, 1]$, and concave revenue curve R_1 with $\hat{q}^\star(R_1) = \hat{q}$. Let $b^\dagger$ be the optimal bid for an agent with value $v(\tilde{q}, R_1)$ ($\triangleq \tilde{v}$) and revenue curve R_1. To show this lemma, it is sufficient to assume $b^\dagger \leq 1/\hat{q}^\star(R_1)$. Now we consider two cases, i.e., $b^\dagger \leq \tilde{v}$ and $b^\dagger \geq \tilde{v}$ separately.

Case (i) $b^\dagger \leq \tilde{v}$: Consider the pentagon revenue curve R_2 with

$$\hat{q}^\star(R_2) = \tilde{q} + \frac{1 - R_1(\tilde{q})}{R_1'(\tilde{q})},\ q_k = \frac{\tilde{q}R_1'(\tilde{q}) - R_1(\tilde{q}) + \frac{R_1(q^\dagger)}{1-q^\dagger}}{R_1'(\tilde{q}) + \frac{R_1(q^\dagger)}{1-q^\dagger}},$$
$$r_k = \frac{1-q_k}{1-q^\dagger} R_1(q^\dagger).$$

where $R_1'(\tilde{q})$ is the right-hand derivative of $R_1(q)$ at $q = \tilde{q}$. By construction, we have (i) $R_2(\tilde{q}) = R_1(\tilde{q})$ and thus $v(\tilde{q}, R_2) = v(\tilde{q}, R_1) = \tilde{v}$; (ii) $R_2(q^\dagger) = R_1(q^\dagger)$; and (iii) $R_2(q) \geq R_1(q)$ for all $q \in [0, q^\dagger]$. See Figure 4.9a for a graphical illustration.

Applying Lemma 4.12 on R_1, R_2, $q^\dagger$, $\tilde{v}$ and all $b^\ddagger \geq b^\dagger$, we conclude that the optimal bid for value $\tilde{v}$ is weakly smaller than $b^\dagger$. Thus, for any bid $b \in [0, 1/\hat{q}]$, if the optimal bid for value $v(\tilde{q}, R_2)$ in revenue curve R_2 is at least b, then the optimal bid $b^\dagger$ for value $v(\tilde{q}, R_1)$ in revenue curve R_1 is at least b as well.

Case (ii) $b^\dagger \geq \tilde{v}$*:* Consider the pentagon revenue curve R_3 with

$$\hat{q}^\star(R_3) = 1 - \frac{1-q^\dagger}{R_1(q^\dagger)},\ q_k = \hat{q}^\star(R_3),\ r_k = 1.$$

By construction, we have (i) $q(\tilde{v}, R_3) \leq q(\tilde{v}, R_1)$ and thus $v(\tilde{q}, R_3) \leq v(\tilde{q}, R_1)$; (ii) $R_3(q^\dagger) = R_1(q^\dagger)$; and (iii) $R_3(q) \geq R_1(q)$ for all $q \in [0, q^\dagger]$. See Figure 4.9b for a graphical illustration.

Applying Lemma 4.12 on R_1, R_3, $q^\dagger$, $\tilde{v}$ and all $b^\ddagger \geq b^\dagger$, we conclude that the optimal bid for value $\tilde{v}$ is weakly smaller than $b^\dagger$. Thus, for any bid $b \in [0, 1/\hat{q}]$, if the optimal bid for value $v(\tilde{q}, R_3)$ in revenue curve R_3 is at least b, then combining with Lemma 4.6, the optimal bid $b^\dagger$ for value $v(\tilde{q}, R_1)$ in revenue curve R_1 is at least b as well. □

Now we are ready to prove Lemma 4.18.

Proof of Lemma 4.18. Fix an arbitrary concave revenue curve R with $\hat{q}^\star(R) \leq 0.62$. We consider $v^*(R) \leq \hat{v}^\star(R)$, $\hat{v}^\star(R) \leq v^*(R) \leq \hat{v}^\star(R)/0.7$, and $v^*(R) \geq \hat{v}^\star(R)/0.7$ separately.

Case (i) $v^*(R) \leq \hat{v}^\star(R)$*:* By Lemma 4.5 and Lemma 4.6, the expected revenue $\mathbf{Rev}_R(\text{SB})$ of the sample-bid mechanism for revenue curve R can be lower-bounded as follows,

$$\begin{aligned}
\mathbf{Rev}_R(\text{SB}) &= \int_0^1 p(v(q,R),R)\,dq \\
&= \int_0^{\hat{q}^\star(R)} p(v(q,R),R)\,dq + \int_{\hat{q}^\star(R)}^{q^*(R)} p(v(q,R),R)\,dq + \int_{q^*(R)}^1 p(v(q,R),R)\,dq \\
&\geq \tilde{p}(\hat{v}^\star(R)/0.7, R)\cdot \hat{q}^\star(R) \\
&\quad + \tilde{p}(\hat{v}^\star(R), R)\cdot (q^*(R) - \hat{q}^\star(R)) + \int_{q^*(R)}^1 p(v(q,R),R)\,dq.
\end{aligned}$$

Invoking Lemma 4.19 and Lemma 4.20, we can rewrite the lower bound of $\mathbf{Rev}[R]$ as

$$\begin{aligned}
&\geq \left(\frac{q^{\ddagger}(R)}{\hat{q}^{\star}(R)} + 0.7w(R) - \frac{0.7\log(\hat{q}^{\star}(R)))}{1-\hat{q}^{\star}(R)}\right)\cdot \hat{q}^{\star}(R) \\
&\quad - \frac{0.7\log(\hat{q}^{\star}(R))}{1-\hat{q}^{\star}(R)} \cdot \left(\frac{1}{1-v^{*}(R)\cdot(1+\hat{q}^{\star}(R))} - \hat{q}^{\star}(R)\right) \\
&\quad + \int_{q^{*}(R)}^{1} p(v(q,R),R)\,dq.
\end{aligned}$$

Note that this lower bound is weakly decreasing in $v^{*}(R)$ while holding everything else fixed. Let $v^{*}(\hat{q}^{\star}(R), q^{\ddagger}(R), w(R))$ be the upper bound of $v^{*}(R)$ as the function of $\hat{q}^{\star}(R)$, $q^{\ddagger}(R), w(R)$ established in Lemma 4.21. From Lemma 4.20, we can lower bound $v^{*}(\hat{q}^{\star}(R),$ $q^{\ddagger}(R), w(R)))$ by $q^{*}(\hat{q}^{\star}(R), q^{\ddagger}(R), w(R)) \triangleq \left(1-v^{*}(\hat{q}^{\star}(R), q^{\ddagger}(R), w(R))\cdot(1+\hat{q}^{\star}(R))\right)^{-1}$. Let $b(q,\hat{q}^{\star}(R))$ be the lower bound of the optimal bid for an agent with value $v(q,R)$ and revenue curve R as the function of $q, \hat{q}^{\star}(R)$ established in Lemma 4.22. Then, we can further rewrite the lower bound of $\mathbf{Rev}[R]$ as

$$\begin{aligned}
&\geq \left(\frac{q^{\ddagger}(R)}{\hat{q}^{\star}(R)} + 0.7w(R) - \frac{0.7\log(\hat{q}^{\star}(R)))}{1-\hat{q}^{\star}(R)}\right)\cdot \hat{q}^{\star}(R) \\
&\quad - \frac{0.7\log(\hat{q}^{\star}(R))}{1-\hat{q}^{\star}(R)} \cdot \left(q^{*}(\hat{q}^{\star}(R), q^{\ddagger}(R), w(R)) - \hat{q}^{\star}(R)\right) \\
&\quad + \int_{q^{*}(\hat{q}^{\star}(R), q^{\ddagger}(R), w(R))}^{1} \frac{0.7\log(b(q,\hat{q}^{\star}(R))\cdot(1-\hat{q}^{\star}(R))+1)}{1-\hat{q}^{\star}(R)}\,dq.
\end{aligned}$$

where the bid $b(q,\hat{q}^{\star}(R))$ in the last term can be lower-bounded using Lemma 4.19.

Therefore, we lower-bound $\mathbf{Rev}_{R}(\mathrm{SB})$ as the function of $\hat{q}^{\star}(R), q^{\ddagger}(R), w(R)$. By numerically enumerating all possible parameters, we conclude that $\mathbf{Rev}_{R}(\mathrm{SB}) \geq 0.545$ in this case.

Case (ii) $\hat{v}^\star(R) \le v^*(R) \le {}^{\hat{v}^\star(R)}/_{0.7}$*:* The analysis is similar to case (i). By Lemma 4.5 and Lemma 4.6, the expected revenue $\mathbf{Rev}_R(\text{SB})$ of the sample-bid mechanism for revenue curve R can be lower-bounded as follows,

$$\begin{aligned}\mathbf{Rev}_R(\text{SB}) &= \int_0^1 p(v(q,R),R)\,dq \\ &\ge \int_0^{q^*(R)} p(v(q,R),R)\,dq + \int_{\hat{q}^\star(R)}^1 p(v(q,R),R)\,dq \\ &\ge \tilde{p}({}^{\hat{v}^\star(R)}/_{0.7}, R) \cdot q^*(R) + \int_{\hat{q}^\star(R)}^1 p(v(q,R),R)\,dq.\end{aligned}$$

Invoking Lemma 4.19 and Lemma 4.20, we can rewrite the lower bound of $\mathbf{Rev}[R]$ as

$$\begin{aligned}&\ge \left(\frac{q^\ddagger(R)}{\hat{q}^\star(R)} + 0.7w(R) - \frac{0.7\log(\hat{q}^\star(R)))}{1-\hat{q}^\star(R)}\right) \cdot \frac{2\hat{q}^\star(R) - q^\ddagger(R)\cdot(1+{}^1/_{0.7})}{1+v^*(R)\cdot(1-\hat{q}^\star(R))} \\ &\quad + \int_{\hat{q}^\star(R)}^1 p(v(q,R),R)\,dq.\end{aligned}$$

Note that this lower bound is weakly decreasing in $v^*(R)$ while holding everything else fixed. Let $v^*(\hat{q}^\star(R), q^\ddagger(R), w(R))$ be the upper bound of $v^*(R)$ established in Lemma 4.21. Let $b(q, \hat{q}^\star(R))$ be the lower bound of the optimal bid for an agent with value $v(q,R)$ and revenue curve R established in Lemma 4.22. Then, we can further rewrite the lower bound as

$$\begin{aligned}&\ge \left(\frac{q^\ddagger(R)}{\hat{q}^\star(R)} + 0.7w(R) - \frac{0.7\log(\hat{q}^\star(R)))}{1-\hat{q}^\star(R)}\right) \\ &\qquad\qquad \cdot \frac{2\hat{q}^\star(R) - q^\ddagger(R)\cdot(1+{}^1/_{0.7})}{1+v^*(\hat{q}^\star(R), q^\ddagger(R), w(R))\cdot(1-\hat{q}^\star(R))} \\ &\quad + \int_{\hat{q}^\star(R)}^1 \frac{0.7\log(b(q,\hat{q}^\star(R))\cdot(1-\hat{q}^\star(R))+1)}{1-\hat{q}^\star(R)}\,dq.\end{aligned}$$

where the bid $b(q, \hat{q}^\star(R))$ in the last term can be lower-bounded using Lemma 4.19.

Therefore, we lower-bound $\mathbf{Rev}_R(\text{SB})$ as the function of $\hat{q}^\star(R), q^\ddagger(R), w(R)$. By numerically enumerating all possible parameters, we conclude that $\mathbf{Rev}_R(\text{SB}) \geq 0.545$ in this case.

Case (iii) $v^*(R) \geq {}^{\hat{v}^\star(R)}/_{0.7}$*:* Lemma 4.21 upper-bounds $v^*(R)$ as the function of $\hat{q}^\star(R)$, $q^\ddagger(R)$ and $w(R)$. By numerically enumerating all possible parameters, we conclude that $v^*(R) \geq {}^{\hat{v}^\star(R)}/_{0.7}$ is not possible for any revenue curve R with $\hat{q}^\star(R) \leq 0.62$. □

4.5. Prior-independent Approximation Lower Bound

In this section, we show that no mechanism can achieve prior-independent approximation better than 1.07 even when the class of distributions are uniform distributions. Note that point mass distributions are special cases of the uniform distributions. The lower bound we will prove in this section holds for more general families of mechanisms than the single-round mechanisms that we introduced in Section 4.1. Here we will show that even when the agent and the seller have multiple rounds of communication in general messages spaces, no mechanism can achieve prior-independent approximation better than 1.07. However, since our analysis does not hinge on the exact format of the mechanism, we will not formally introduce the model for multi-rounds of communication.

Theorem 4.23. *For a single item, a single uniformly distributed agent, and a single valuation sample, the prior-independent approximation ratio for revenue maximization is at least* 1.07*.*

The main idea for proving Theorem 4.23 is as follows. Consider two scenarios where the valuation distribution of the agent is either uniform between $[1, 2]$ or a pointmass with some value $v \in [1,2]$. Note that the optimal mechanism for an agent with value from the uniform distribution U$[1,2]$ is to always allocate the item with expected payment 1. Thus if the mechanism is optimal for this setting, when the valuation distribution for the agent is actually a pointmass with some value $v \in [1,2]$, the agent can always imitate the type in a uniform distribution $U[1,2]$ to win the item and pay at most 1 in expectation. This indicates that the optimal prior-independent approximation ratio is strictly above 1. By leveraging the approximation ratio in those two cases, we show that the optimal ratio is at least 1.07.

Before the proof of Theorem 4.23, we first introduce several notations and present several properties for non-truthful mechanisms $\mathcal{M}$ with prior-independent approximation ratio β.

Lemma 4.24. *For single item, single agent, any distribution F with support $[\underline{v}, \overline{v}]$, for non-truthful mechanism with prior-independent approximation ratio β, the interim allocation for agent with highest value $\overline{v}$ is $x(\overline{v}, F) \geq \frac{1}{\beta}$.*

Proof. Suppose the interim allocation for agent with value $\overline{v}$ is $x(\overline{v}, F) < \frac{1}{\beta}$. Since the interim allocation is monotone, the maximum expected virtual welfare for mechanism under distribution F is less than $1/\beta$ of the optimal expected virtual welfare, which implies the revenue is less than $1/\beta$ of the optimal revenue and the approximation ratio for distribution F is higher than β, a contradiction. $\square$

Lemma 4.25. *For single item, single agent, and any uniform distribution F with support $[\underline{v}, \overline{v}]$ such that $2\underline{v} \geq \overline{v}$, for a non-truthful mechanism with prior-independent approximation ratio β, the interim utility for agent with highest value $\overline{v}$ is $u(\overline{v}, F) \geq \frac{1}{2}\left(\overline{v} - \sqrt{\overline{v}^2 - \frac{4\underline{v}}{\beta}(\overline{v} - \underline{v})}\right)$.*

Proof. For uniform distribution F with support $[\underline{v}, \overline{v}]$ such that $2\underline{v} \geq \overline{v}$, the optimal mechanism OPT_F is to post price $\underline{v}$ with expected revenue $\underline{v}$. Suppose the utility for agent with value $\overline{v}$ is $u(\overline{v}, F) < \frac{1}{2}\left(\overline{v} - \sqrt{\overline{v}^2 - \frac{4\underline{v}}{\beta}(\overline{v} - \underline{v})}\right)$, the optimal mechanism subject to this constraint is to post price $\overline{v} - u(\overline{v}, F)$, with expected revenue $\frac{u(\overline{v},F)}{\overline{v}-\underline{v}} \cdot (\overline{v} - u(\overline{v}, F)) < \frac{\underline{v}}{\beta}$, a contradiction. $\square$

Lemma 4.26. *For single item, single agent, any point mass distribution F with support $\overline{v}$, for non-truthful mechanism with prior-independent approximation ratio β, the interim utility for agent with value $\overline{v}$ is $u(\overline{v}, F) \leq \overline{v}(1 - 1/\beta)$.*

Proof. Suppose the interim utility in this case is $u(\overline{v}, F) > \overline{v}(1 - 1/\beta)$, the expected revenue is at most the social welfare minus the expected utility, which is at most $\overline{v} - u(\overline{v}, F) < \frac{\overline{v}}{\beta}$, contradicting the fact that mechanism $\mathcal{M}$ achieves prior-independent approximation ratio β. $\square$

Proof of Theorem 4.23. Suppose mechanism $\mathcal{M}$ inducing interim allocation and payment rule x and p achieves prior-independent approximation ratio β. Consider uniform distribution F with support $[1, 2]$. By Lemma 4.24 and 4.25, we have $x(2, F) \geq \frac{1}{\beta}$, and $u(2, F) \geq 1 - \sqrt{1 - 1/\beta})$. For any sample $s \in [1, 2]$, the expected allocation and payment of agent with value 2 given the sample s satisfies the constraint that

$$s \cdot x(2, F, s) - p(2, F, s) \leq s\left(1 - \frac{1}{\beta}\right) \tag{4.3}$$

otherwise for distribution F_s with point mass on s, an agent with value s can imitate the behavior of an agent with value 2 in uniform distribution to achieve utility strictly higher than $s\,(1 - 1/\beta)$, and by Lemma 4.26, this contradicts to the assumption that mechanism

$\mathcal{M}$ achieves prior-independent approximation ratio β. Taking expectation over sample s for the left hand side of equation (4.3), we have

$$\begin{aligned}&\mathbf{E}_s[s \cdot x(2, F, s) - p(2, F, s)] \geq \mathbf{E}_s[s \cdot x(2, F, s)] - (2 - u(2, F))\\ &\geq \int_1^{1+1/\beta} s \; ds - (2 - u(2, F))\end{aligned}$$

where the last inequality holds because $x(2, F) \geq \frac{1}{\beta}$ and the worst case happens when $x(2, F, s) = 0$ for any sample $s \geq 1 + 1/\beta$. Taking expectation over sample s for the right hand side of equation (4.3), we have

$$\mathbf{E}_s\left[s\left(1 - \frac{1}{\beta}\right)\right] = \frac{3}{2}\left(1 - \frac{1}{\beta}\right).$$

Combining the inequalities, we have

$$\frac{1}{2}\left(1 + \frac{1}{\beta}\right)^2 - \frac{1}{2} - (1 + \sqrt{1 - 1/\beta}) \leq \frac{3}{2}\left(1 - \frac{1}{\beta}\right).$$

By solving the inequality, we have $\beta \geq 1.0737$. $\square$

4.6. Revenue of the Sample-based Pricing

Allouah and Besbes (2019) characterized the prior-independent approximation ratio of the truthful mechanisms under the assumption of scale-invariance for sample-based pricing mechanisms. Note that in contrast, our lower bound result shown in Theorem 4.23 does not require the assumption on scale-invariance. Here is the formal definition of sample-based pricing mechanisms.

Definition 4.10. *Given function $\alpha : \mathbb{R} \to \Delta(\mathbb{R})$ mapping from the sample to the randomized price, for sample s, the* sample-based pricing mechanism *solicits a non-negative bid $b \geq 0$, allocates the item to the agent if $b \geq \alpha(s)$, and charges the agent $\alpha(s) \cdot \mathbb{1}\{b \geq \alpha(s)\}$.*

It can be observed that the bid allocation rules of both sample-bid mechanism and sample-based pricing are similar (i.e. competing against the sample), and the difference is the payment semantics.

Theorem 4.27 (Allouah and Besbes, 2019)**.** *Under the assumption of scale-invariance, for single-item setting with regular valuation distribution, when seller has access to a single sample, the prior-independent approximation ratio of the optimal sample-based pricing mechanism is bounded in* $[1.957, 1.996]$*. Moreover, when the valuation distribution is MHR, the prior-independent approximation ratio is bounded in* $[1.543, 1.575]$*.*

4.7. Revelation Gap for Revenue Maximization

To establish our final revelation gap, we impose a further restriction (i.e., scale-invariant) to the class of revelation mechanisms.

Definition 4.11. *A mechanism is* scale-invariant *if the interim allocation is invariant of the scale, i.e., $x(\alpha v, \alpha F) = x(v, F)$ for any distribution F, valuation v and any $\alpha > 0$.*

Given an arbitrary valuation distribution and any mechanism that is incentive compatible only for the given valuation distribution, the mechanism may not be equivalent to any sample-based pricing mechanism. The is because the agent only maximizes her utility by taking expectation over the sample. However, we can show that if the mechanism is incentive compatible for all possible prior distributions, then it is equivalent to consider posting a randomized price to the agent based on the realization of the sample, i.e., a sample-based pricing mechanism.

Lemma 4.28. *For any mechanism with allocation $\tilde{x}$ and payment $\tilde{p}$ that is incentive compatible and individual rational for all valuation distributions, there exists a sample-based pricing mechanism that generates the same expected allocation and payment pointwise for any valuation of the agent and any realization of the sample.*

Proof. First we claim that, for any truthful mechanism with allocation $\tilde{x}$ and payment $\tilde{p}$, the induced allocation rule $\tilde{x}(\cdot, s)$ and payment rule $\tilde{p}(\cdot, s)$ are incentive compatible and individual rational given any realization of the sample s.

First we prove the incentive compatibility. Suppose by contradiction, there exists constant $\epsilon > 0$, sample s and value v, v' such that

$$v\tilde{x}(v', s) - \tilde{p}(v', s) \geq v\tilde{x}(v, s) - \tilde{p}(v, s) + \epsilon.$$

Let F be an arbitrary distribution with positive density everywhere on the support $[0, \infty)$. Define $H \triangleq u(v, v, F) - u(v, v', F)$ as the utility loss for value v to misreport v' when the distribution is F. Given constant $\delta > 0$, let F' be the distribution such that with probability $1-\delta$, the value of the agent is s and with probability δ, the value is drawn from distribution F. It is easy to verify that both v and v' are in the support of distribution F'. Moreover, the utility loss for misreporting v' is

$$u(v, v, F') - u(v, v', F') \geq (1 - \delta)\epsilon + \delta H$$

where $(1-\delta)\epsilon + \delta H > 0$ for sufficiently small δ. This implies that the mechanism is not incentive compatible for distribution F', a contradiction.

Similarly, for individual rationality, if there exists constant $\epsilon > 0$, sample s and value v, v' such that

$$v\tilde{x}(v, s) - \tilde{p}(v, s) \leq -\epsilon,$$

there exists a distribution F' supported on $[0, \infty)$ such that agent with value v is not individual rational given distribution F'.

Finally, since for any sample s, the induced mechanism is incentive compatible, the allocation $\tilde{x}(v, s)$ is monotone in v for any sample s. Moreover, individual rationality implies that the payment of the agent is 0 if she does not win the item. Thus the mechanism can be implemented as sample-based pricing mechanism for any realized sample. $\square$

Lemma 4.28 suggest that under the assumption of scale invariance, the bounds on prior-independent approximation ratio of sample-based pricing in Theorem 4.27 carry over to truthful mechanisms. Then combining it with Theorem 4.11 and 4.23, we have the following corollary characterizing the revelation gap under the assumption of scale-invariance.

Corollary 4.29. *Under the assumption of scale-invariance, for single-item setting with regular valuation distribution, when seller has access to a single sample, the revelation gap is bounded in* $[1.066, 1.859]$*. Moreover, when the valuation distribution is MHR, the revelation gap is bounded in* $[1.190, 1.467]$*.*

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APPENDIX A

Appendix to Chapter 3

A.1. Clinching Auction with Price Jumps

In this section, we introduce the clinching auction with price jumps. In the standard clinching auction, with a continuous increasing price-clock, excess demand decreases continuously to the point where supply equals demand and the market clears. With a price jump, which leads to a strict drop of demands, the standard clinching auction may leave some supply unallocated. Therefore, to clear the market, the clinching auction with price jumps will need to reallocate some amount of units at the pre-jump price after a price jump. We first focus on the clinching auction with price jumps for agents with identical budgets. The results can be extended to agents with distinct budgets, which we will discuss at the end of this section.

To formally describe this reallocation, suppose the price-clock jumps from $v^\dagger$ to $v^\ddagger$. Consider the state $\mathcal{C} = (s, S^\dagger, w)$ at price $v^\dagger$ after the clinching step (i.e. Step 3 in Definition 3.4) where s is the current supply remaining, $S^\dagger$ is the agents with values at least $v^\dagger$ (let $k^\dagger = |S^\dagger|$), and w is the current budget of the active agents. [1] When the price jumps to $v^\ddagger$, active agents with values below $v^\ddagger$ ("low-valued" agents) will quit, while active agents with values at least $v^\ddagger$ ("high-valued" agents) will stay in the auction. Denote by $S^\ddagger$ the

[1]In the model considered in this paper where initially the agents have identical budgets, the remaining budgets of all active agents remain identical throughout the execution of the clinching auction.

set of high-valued agents and by $k = |S^\ddagger|$ the number of high-valued agents. With pre-jump state $\mathcal{C} = (s, S^\dagger, w)$ and k agents remaining after the jump, define $h_k^{\mathcal{C}}$ and $l_k^{\mathcal{C}}$ as the additional supply allocated at the low price to high- and low-valued agents, respectively.

In the following discussion, we fix an arbitrary state $\mathcal{C} = (s, S^\dagger, w)$ with $k^\dagger = |S^\dagger|$ active agents, drop the superscript of $h_k^{\mathcal{C}}$ and $l_k^{\mathcal{C}}$, and consider h_k and l_k constrained to the following polytope:

$$
\begin{aligned}
&\text{IC: } \forall k \in \{1, \ldots, k^\dagger\} && h_k = l_{k-1}, \\
&\text{BB: } \forall k \in \{0, \ldots, k^\dagger\} && h_k, l_k \le w/v^\dagger, \\
&\text{NN: } \forall k \in \{0, \ldots, k^\dagger\} && h_k, l_k \ge 0, \\
&\text{MC: } \forall k \in \{0, \ldots, k^\dagger\} && k\, h_k + (k^\dagger - k)\, l_k + \tfrac{k}{v^\ddagger}(w - v^\dagger h_k) \ge s, \\
&\text{LS: } \forall k \in \{0, \ldots, k^\dagger\} && k\, h_k + (k^\dagger - k)\, l_k \le s.
\end{aligned}
\tag{A.1}
$$

The constraints above are, respectively, incentive compatibility (IC), budget balance (BB), non-negative consumption (NN), market clearing (MC), and limited supply (LS). The IC constraint requires that the amount of supply which an agent gets at price $v^\dagger$ does not depend on whether the agent stays or quits during the price jump. The left-hand side of the incentive compatibility (IC) constraint is the additional allocation quantity at price $v^\dagger$ if an agent stays during the price jump, while the right hand side is the additional allocation quantity at price $v^\dagger$ if she quits. Since the two quantities are equal, active agents with values in $[v^\dagger, v^\ddagger)$ prefer to quit after the clinching step at price $v^\dagger$ while agents with value at least $v^\ddagger$ prefer to stay in the auction. The market clearing (MC) constraint states that the reallocated supply at the low price $v^\dagger$ (the first two terms) plus the quantity demanded by the high-valued agents at the high price $v^\ddagger$ (the third term) must be at least the supply. The limited supply (LS) constraint states that the amount allocated at the low price for any number k of high-valued agents must not exceed the supply.

Consider the problem of selecting a point in polytope (A.1) to optimize the expected welfare under the value distribution F. First notice that, since the state $\mathcal{C} = (s, S^\dagger, w)$ is induced by the clinching auction, the total demand at price $v^\dagger$ under budget w exceeds the remaining supply s, i.e., $k^\dagger w/v^\dagger \geq s$; setting $h_k = l_k = s/k^\dagger$ for all $k \in [k^\dagger]$ is feasible; and, thus, polytope (A.1) is not empty. The expected welfare of the clinching auction is complicated to express; we instead consider the objective of minimizing, within the constraints of polytope (A.1), the expected supply reallocated to low-valued agents, i.e., $\sum_{k=0}^{k^\dagger} l_k \pi^{k^\dagger - k}(1-\pi)^k$ where $\pi = \frac{F(v^\ddagger)-F(v^\dagger)}{1-F(v^\dagger)} = \mathbf{Pr}_{v\sim F}\left[v < v^\ddagger \mid v \geq v^\dagger\right]$ is the probability an agent has a low value. Based on this reallocation, we formally define a clinching auction with price jumps and show that it clears the market, is ex-post IR, and is DSIC.

Definition A.1. *The* clinching auction with price jumps *maintains an allocation and price-clock starting from zero. Before and after each price jump point, the price-clock ascends continuously and the allocation and the budget are adjusted as in the standard clinching auction. When the price-clock jumps from $v^\dagger$ to $v^\ddagger$ the following steps are taken:*

(1) run the standard clinching steps on price-clock $v^\dagger$ and the current budgets and let the subsequent state be $\mathcal{C} = (s, S^\dagger, w)$ with $k^\dagger = |S^\dagger|$;

(2) increase the price-clock to $v^\ddagger$ and let $k^\ddagger = |S^\ddagger|$ be the number of agents remaining in the auction;

(3) solve for $\{h_k, l_k\}_{k\in[k^\dagger]}$ to minimize the expected quantity reallocated to the low-valued agents in the polytope (A.1)*;*

(4) allocate $h_{k^\ddagger}$ *units at price* $v^\dagger$ *to each of the* $k^\ddagger$ *agents that stay after the price jump, allocate* $l_{k^\ddagger}$ *units at price* $v^\dagger$ *to each of the* $k^\dagger - k^\ddagger$ *agents that quit during the price jump, and adjust all the agents' budgets for the amount and price allocated;*

(5) run the standard clinching step with price-clock $v^\ddagger$ *and updated budgets.*

Proposition A.1. *The clinching auction with price jumps always clears the market.*

Proof. If the price-clock increases continuously, the demand decreases continuously. When the total demands meet the supply remaining, Dobzinski et al. (2008) show that the standard clinching auction halts and the market clears. For the clinching auction with price jumps, when the price-clock goes through a price jump, the market clearing constraints that define polytope (A.1) guarantee that the total demands are at least the supply remaining. Thus, the clinching auction with price jumps clears the market. □

Proposition A.2. *The clinching auction with price jumps satisfies ex-post IR, DSIC, and budget balance.*

Proof. Dobzinski et al. (2008) show that the standard clinching auction is ex-post IR, DSIC, and budget balanced. For the clinching auction with price jumps, when the price-clock goes through a price jump from $v^\dagger$ to $v^\ddagger$, the IC constraints that define polytope (A.1) guarantee that the agents with values at most $v^\ddagger$ weakly prefer to quit at price $v^\dagger$ and the agents with values above $v^\ddagger$ prefer to stay at price $v^\dagger$. Meanwhile, the budget constraints and non-negative consumption constraints that define polytope (A.1) guarantee that the agents are budget balanced and have non-negative utility after the price jump. □

For two i.i.d. agents with identical budgets, the clinching auction with price jumps induces the same outcome as the middle-ironed clinching auction (Definition 3.5). For a general number of agents, it is polynomial time solvable. We conjecture that, for i.i.d. distributions and identical budgets, minimizing the expected quantity reallocated to low-valued agents, i.e., the objective described previously, is equivalent to maximizing expected welfare. We leave to future studies the question of whether there is a more succinct characterization of the expected welfare maximizing solution and the generalization to agents with non-identical valuation distributions.

If agents have distinct budgets, the linear program can be generalized by replacing the variables, which corresponded to the reallocation to high- and low-valued agents with a given number $k = |S^{\ddagger}|$ of high-valued agents, with variables that correspond to the reallocation to each agent i with a given set $S^{\ddagger}$ of high-valued agents. With this modification to the variables and constraints of polytope (A.1), the previous argument guarantees the new polytope is non-empty. Notice that there are $O(n \cdot 2^n)$ variables defining the new polytope. It is possible, however, to optimize expected allocation to the low-valued agents subject to this polytope in polynomial time when there are a constant number of distinct budgets; symmetries across agents with identical budgets allow the number of variables in the program to be reduced to a polynomial number. We leave to future studies the problem of identifying a polynomial time algorithm for optimally reallocating the supply during a price jump when there are generally distinct budgets.

APPENDIX B

Appendix to Chapter 4

B.1. Numerical Analysis for the Sample-bid Mechanism

In Section 4.4, we bound the prior-independent approximation ratio of the sample-bid mechanism by enumerating the possible choices of given parameters. One concern is that the parameters are selected from a continuous interval, and the revenue for valuation distributions with parameters that are not evaluated on discretized points may be far from the revenue on discretized points. In this section, we formally show that this is not the case for our analysis. To provide a theoretical lower bound on all possible distributions, we will present a unified lower bound on the revenue for distributions with parameters between discretized points. We will formalize this approach for the numerical calculation for Lemma 4.17, and the numerical calculation for other lemmas and theorems hold similarly.

By the proof of Lemma 4.17, for any revenue curve R in Figure 4.5b parameterized by monopoly quantile $\hat{q}^\star \in [\underline{q}_m, \bar{q}_m]$ and revenue $r_0 \in [\underline{r}_0, \bar{r}_0]$ for quantile 0, the revenue of the seller is lower bounded by $p(v^*(R), R) \cdot q^*(R)$ where $v^*(R)$ is the critical value with bid above monopoly price and $q^*(R)$ is the quantile for critical value. Note that it is sufficient for us to consider revenue curves R such that $v^*(R)$ is at least the monopoly price. Next we show how to provide bounds on parameters $\underline{q}_m, \bar{q}_m, \underline{r}_0, \bar{r}_0$, as well as lower bounds on $p(v^*(R), R)$ and $q^*(R)$ using parameters $\underline{q}_m, \bar{q}_m, \underline{r}_0, \bar{r}_0$.

Lemma B.1. *There exists efficiently computed set $S \subseteq \mathbb{R}^4$ and function $\tau : \mathbb{R}^4 \to \mathbb{R}$ such that for any revenue curve R in Figure 4.5b parameterized by monopoly quantile $\hat{q}^\star \in [\underline{q}_m, \bar{q}_m]$ and revenue $r_0 \in [\underline{r}_0, \bar{r}_0]$ for quantile 0, we have*

(1) $v^*(R) \geq \hat{v}^\star(R)$ only if $(\underline{q}_m, \bar{q}_m, \underline{r}_0, \bar{r}_0) \in S$;

(2) $p(v^*(R), R) \cdot q^*(R) \geq \tau(\underline{q}_m, \bar{q}_m, \underline{r}_0, \bar{r}_0)$ if $(\underline{q}_m, \bar{q}_m, \underline{r}_0, \bar{r}_0) \in S$.

Proof. First we illustrate how to find the desirable set S by numerical calculation. Note that the requirement is such that the critical value for bidding above the monopoly price is above monopoly price, i.e., $v^*(R) \geq \hat{v}^\star(R)$. By Lemma 4.16, it is sufficient to verify that the optimal utility of value $\hat{v}^\star(R)$ for bidding above $\hat{v}^\star(R)$ is positive. Note that by Lemma 4.3, the optimal bid above the monopoly price is $b = \frac{\hat{v}^\star}{\alpha} + \frac{1-r_0}{1-\hat{q}^\star}$, with expected utility

$$u(\hat{v}^\star, b) = \frac{1}{\hat{q}^\star} \cdot (1 - q_b) - \alpha \left(b \cdot q_b + r_0 \log(\frac{\hat{q}^\star}{q_b}) + \frac{(1-r_0)(\hat{q}^\star - q_b)}{\hat{q}^\star} - \log \hat{q}^\star \right)$$

where $q_b = \frac{r_0}{b - \frac{1-r_0}{\hat{q}^\star}}$. Since $\hat{q}^\star \in [\underline{q}_m, \bar{q}_m]$ and $r_0 \in [\underline{r}_0, \bar{r}_0]$, a sufficient condition for $u(\hat{v}^\star, b) > 0$ is that

$$\frac{1}{\bar{q}_m} \cdot (1 - \bar{q}_b) - \alpha \left(\bar{b} \cdot \bar{q}_b + \bar{r}_0 \log(\frac{\bar{q}_m}{\underline{q}_b}) + \frac{(1-\underline{r}_0)(\bar{q}_m - \underline{q}_b)}{\underline{q}_m} - \log \underline{q}_m \right) > 0,$$

where $\bar{b} = \frac{1}{\alpha \underline{q}_m} + \frac{1-\underline{r}_0}{1-\bar{q}_m}$, $\bar{q}_b = \frac{\alpha \bar{r}_0 \bar{q}_m (1-\underline{q}_m)}{1-\bar{q}_m+\alpha(1-\bar{r}_0)}$ and $\underline{q}_b = \frac{\alpha \underline{r}_0 \underline{q}_m (1-\bar{q}_m)}{1-\underline{q}_m+\alpha(1-\underline{r}_0)}$. Note that the above inequality can be easily verified on discretized points.

Next we construct the function $\tau(\underline{q}_m, \bar{q}_m, \underline{r}_0, \bar{r}_0)$ lower bound the revenue $p(v^*(R), R) \cdot q^*(R)$. First note that we can enumerate the value above monopoly price and find the minimum value that the interim utility is strictly positive. That is, given value $v \geq \hat{v}^\star$, the optimal bid above the monopoly price is $b = \frac{v}{\alpha} + \frac{1-r_0}{1-\hat{q}^\star}$, with expected utility

$$
\begin{aligned}
u(v,b) &= v \cdot (1-q_b) - \alpha \left(b \cdot q_b + r_0 \log(\frac{\hat{q}^\star}{q_b}) + \frac{(1-r_0)(\hat{q}^\star - q_b)}{\hat{q}^\star} - \log \hat{q}^\star \right) \\
&\geq v \cdot (1-\bar{q}_b) - \alpha \left(\bar{b} \cdot \bar{q}_b + \bar{r}_0 \log(\frac{\bar{q}_m}{\underline{q}_b}) + \frac{(1-\underline{r}_0)(\bar{q}_m - \underline{q}_b)}{\underline{q}_m} - \log \underline{q}_m \right) > 0,
\end{aligned}
$$

where $\bar{b} = \frac{v}{\alpha} + \frac{1-\underline{r}_0}{1-\bar{q}_m}$, $\bar{q}_b = \frac{\alpha \bar{r}_0 \bar{q}_m (1-\underline{q}_m)}{v\underline{q}_m(1-\bar{q}_m)+\alpha(1-\bar{r}_0)}$ and $\underline{q}_b = \frac{\alpha \underline{r}_0 \underline{q}_m (1-\bar{q}_m)}{v\bar{q}_m(1-\underline{q}_m)+\alpha(1-\underline{r}_0)}$. Let v^* be the minimum value that satisfies the above inequality. Then we have $v^* \geq v^*(R)$, and hence

$$
q^*(R) \geq q(v^*, R) = \frac{\alpha r_0 \hat{q}^\star (1-\hat{q}^\star)}{v\hat{q}^\star(1-\hat{q}^\star) + \alpha(1-r_0)} \geq \frac{\alpha \underline{r}_0 \underline{q}_m (1-\bar{q}_m)}{v\bar{q}_m(1-\underline{q}_m) + \alpha(1-\underline{r}_0)}.
$$

Moreover, we can similar construct an upper bound on the utility $u(v,b)$ and let $\underline{v}^*$ be the largest value such that the upper bound on the utility is at most 0. Thus, we have $v^*(R) \geq \underline{v}^*$ and hence

$$
\begin{aligned}
p(v^*(R), R) \geq p(\underline{v}^*, R) &= \alpha \left(b \cdot q_b + r_0 \log(\frac{\hat{q}^\star}{q_b}) + \frac{(1-r_0)(\hat{q}^\star - q_b)}{\hat{q}^\star} - \log \hat{q}^\star \right) \\
&\geq \alpha \left(\underline{b} \cdot \underline{q}_b + \underline{r}_0 \log(\frac{\underline{q}_m}{\bar{q}_b}) + \frac{(1-\bar{r}_0)(\underline{q}_m - \bar{q}_b)}{\bar{q}_m} - \log \bar{q}_m \right)
\end{aligned}
$$

where $\underline{b} = \frac{\underline{v}^*}{\alpha} + \frac{1-\bar{r}_0}{1-\underline{q}_m}$, $\bar{q}_b = \frac{\alpha \bar{r}_0 \bar{q}_m (1-\underline{q}_m)}{\underline{v}^* \underline{q}_m(1-\bar{q}_m)+\alpha(1-\bar{r}_0)}$ and $\underline{q}_b = \frac{\alpha \underline{r}_0 \underline{q}_m (1-\bar{q}_m)}{\underline{v}^* \bar{q}_m(1-\underline{q}_m)+\alpha(1-\underline{r}_0)}$. By combining the inequalities, we have an lower bound on $p(v^*(R), R) \cdot q^*(R)$ as a function of $(\underline{q}_m, \bar{q}_m, \underline{r}_0, \bar{r}_0)$. By discretizing the feasible set and enumerating for all discretized points, we have a unified lower bound on revenue for all possible distributions. $\square$