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Abstract

This dissertation consists of four essays that analyze strategic outcomes in economic environments. The first three essays use network theory as a tool to study optimal interventions in economic environments. The fourth essay studies welfare outcomes in multi-unit first-price auctions.

The first essay is co-authored with Krishna Dasaratha and Benjamin Golub. A principal uses payments conditioned on stochastic outcomes of a team project to elicit costly effort from the team members. We develop a multi-agent generalization of a classic first-order approach to contract optimization by leveraging methods from network games. The main results characterize the optimal allocation of incentive pay across agents and outcomes. Incentive optimality requires equalizing, across agents, a product of (i) individual productivity (ii) organizational centrality and (iii) responsiveness to monetary incentives. We specialize the model to explore several applied questions, including whether compensation should reward individual ability or collaborativeness and how the strength of complementarities shapes pay dispersion.

The second essay is co-authored with Krishna Dasaratha and Ben Golub. In this essay, we further study properties of optimal contracts in a linear-quadratic parametric environment. We study the problem of selecting the team of agents who receive positive pay, and show this team must form a tight-knit subset of the complementarity network, with any pair being complementary to one another or jointly to another team member. Finally, we give conditions under which the amount of payment used for compensation is increasing in the strength of a team's complementarities and discuss several other applications.

The third essay is co-authored with Krishna Dasaratha. Consider a network game with linear best responses and spillovers between players, and let agents endogenously choose their links. A planner considers interventions to subsidize actions and/or links between players, aiming to maximize a welfare objective depending on equilibrium actions. The structure of an optimal intervention is shaped by the intrinsic value that links provide to agents. When this value is non-negative, it is optimal to focus only on subsidizing actions. When it is negative,

we give conditions under which it is necessary to include link subsidies. This reverses the basic structure of the optimal intervention in settings with exogenous links.

The fourth essay is co-authored with Jason Hartline and Matthew vonAllmen. This paper performs a robust analysis of welfare in multi-unit first-price auctions with independent private values. We characterize auction performance by the worst-case loss in welfare over all ϵ -approximate equilibria. The central object of our analysis is the minimum selling price of units. We derive a measure of uncertainty for the minimum price and show how this translates into inefficiencies for large supply. Under Bayes-Nash equilibrium, the minimum selling price is deterministic in the limit and there is no loss in efficiency. This recovers a classical result from the literature. Under approximate equilibria, we characterize the worst-case limit distribution of the minimum selling price and establish a bound on inefficiencies. We also derive key structural properties of approximate and Bayes-Nash equilibria — results that may be of independent interest.

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Chapter 1

Incentive Design With Spillovers

1.1 Introduction

A popular method of motivating members of a team is giving them performance incentives that depend on jointly achieved outcomes. Examples of such incentives include startup executives receiving firm stock and a marketing team receiving bonuses for achieving a sales target. How should such incentive schemes be designed and how should they take into account the team's production function?

We examine these questions in a simple model of a team working on a joint project. Each member chooses how much costly effort to exert. These actions jointly determine a real-valued *team performance*—for example, the quality of a product—according to a sufficiently smooth, increasing function of the efforts, which may entail interactions such as complementarities among agents' efforts. Any performance level determines a probability distribution over observable project *outcomes*. For example, the outcome may be the revenue from a project, which is increasing in non-contractible project quality, but stochastic due to factors outside the team's control. Although it is not possible to write contracts contingent on individual actions or the realized team performance, the principal can commit to a contract specifying nonnegative payments to each agent contingent on each possible project outcome. The principal's goal is to design this contract in a way that maximizes profit: revenue minus

compensation.

The setting builds on the classic [Holmström \[1979\]](#) model, in which a single agent produces work of a non-contractible quality resulting in an observable outcome.¹ In our setting, the analogue of quality is a jointly achieved performance. How incentive spillovers across agents matter for contract design—a central issue for modern firms—is not well understood, despite the immense amount learned about contract design since Holmström’s work. In this paper, we make progress on this problem by leveraging some ideas from network theory.

To illustrate the basic importance of incentive spillovers, imagine that the principal slightly adjusts the contract of a particular agent, Bob, in a way that motivates him to work harder. In team production, changing one team member’s action can change other agents’ private returns to effort, holding fixed their own contracts. Those whose efforts are complements to Bob’s are now motivated to work harder, while those whose efforts are substitutes to his have incentives to free-ride on his higher effort. Optimal contracts therefore depend on both the direct effect of each agent’s action on team performance and the responses that action induces from others.

The paper makes two contributions: first, it provides a general characterization of optimal contracts in the presence of incentive spillovers; second, it applies this framework to understand how exogenous features of production in an organization shape optimal compensation outcomes.

The condition that constitutes the first contribution is stated in terms of three kinds of quantities that can be associated to each agent at any contract. The first quantity, an agent’s *marginal productivity*, is the partial derivative of team performance in an individual’s action, holding others’ actions fixed. The second quantity is called an agent’s *centrality*: a measure of connectedness in a network—which we are about to discuss in detail—reflecting incentive spillovers, with connectedness to more productive agents weighted more. The relevance of these quantities for contract optimality should be intuitive in view of what we have said. The third quantity is an agent’s *responsiveness* to additional incentives, which involves the marginal utility of money and the curvature of the agent’s cost. It accounts for the fact that

¹In particular, the obstacles to perfect contracting are the same: moral hazard and limited liability.

an agent who has a low valuation of an additional dollar and finds it considerably more costly to take a larger action is, all else equal, a less appealing recipient of incentive pay. Our main result is that, when the binding incentive constraints are local, optimal contracts satisfy a balance condition: the product of the three quantities described above is equal across all agents receiving any incentive pay.

The spillover matrix used to define centrality plays a key role in our analysis. Economically, it summarizes how changing one agent’s effort shifts other agents’ marginal returns to effort, inducing strategic responses that propagate through the team: complements work harder, while substitutes have incentives to free-ride. Mathematically, the network is based on the Hessian of the team performance function evaluated at the equilibrium induced by the optimal contract—that is, the cross-partial derivatives of output as we differentiate in two agents’ efforts. Because this network is pinned down by the technology’s local curvature, it can in principle be estimated from data on how output changes as agents’ actions change. It is worth emphasizing that our general model makes no parametric assumptions and thus allows quite flexible production functions for the team.² Nevertheless, we show that the team’s production function matters only through its first derivatives and its Hessian at the optimal contract. This allows us to leverage methods from well-understood network games whose payoffs are quadratic polynomials to analyze the spillover effects of locally perturbing incentive pay under arbitrary contracts. We use this key reduction to characterize the first-order conditions that determine the principal’s optimal allocation of incentives both across agents and across outcomes.

There are two immediate corollaries of this characterization, new to the contract theory literature, that demonstrate its economic content. First, it implies a ranking of agents by compensation. If all agents have identical utility functions of money and these functions are concave, then those with a higher “productivity times centrality” index must be paid more in every outcome. This ranking favors those agents who have high marginal productivity themselves and whose actions are also valuable to others’ incentives directly and indirectly. Second, the balance condition implies a ranking of observable *outcomes* (e.g., revenue real-

²To take just one example, the team’s production function could be an arbitrary polynomial, with monomials of arbitrary degree reflecting outputs generated through the joint efforts of arbitrary subsets of the team—e.g., complementing one another in threes, fours, and so on.

izations) based on their incentive power, and this ranking holds across agents. Payments concentrate on outcomes with higher “incentive power”—those where the ratio of probability to marginal probability is favorable for motivating effort. This generalizes the single-agent result of [Holmström \[1979\]](#) to our multi-agent setting.

The balance condition is a critical step in our analysis of the problem, but all its terms are endogenous—evaluated at an equilibrium played by the agents at a contract chosen by the principal. To obtain more direct guidance, one needs to understand how exogenous features of the organization—its technological complementarities and measurable agent characteristics—map into compensation outcomes. The remaining sections take up this challenge by introducing several different parametric structures that admit explicit solutions, including standard network games settings and CES production.

We begin in [Section 1.4.1](#) with a setting where agents differ in the contribution their effort makes to performance in isolation, which we call *standalone productivity*, and in how much their effort complements others’, with each agent’s complementarity described by a single parameter. In this tractable case, the balance condition yields explicit pay formulas: optimal compensation depends on a combination of an agent’s standalone productivity and his complementarity parameter. As spillovers strengthen, compensation shifts sharply from being standalone-productivity-driven to being complementarity-driven. A numerical example suggests accounting for this effect can be very important: a principal neglecting spillovers when they are large fails to achieve most of the potential profits.

The rank-one case used to make these points has the property that an exogenous complementarity parameter for each agent maps monotonically into that agent’s centrality. The relation is more complex with general complementarity networks. [Section 1.4.2](#) works in a setting with equal standalone productivities and gives a closed-form characterization of centralities and optimal compensation. Centrality need not be monotone in complementarity parameters: strengthening complementarities can weaken an agent’s network position and pay because the principal rebalances the optimal contract and equilibrium actions.

Our applications so far have been set in the linear-quadratic framework, which is standard in network games but in which all complementarities are structurally bilateral. A different case of practical interest is one in which complementarities arise at the level of the whole

team rather than between pairs. Motivated by this, [Section 1.5](#) specializes the framework to Cobb–Douglas and CES production functions capturing this feature. There, we show that the balance condition again yields clean closed-form solutions. This allows us to address an important outcome: pay dispersion. The key insight is that pay dispersion is governed by the elasticity of substitution: highly substitutable efforts justify concentrating pay on high-productivity agents, while strong team-level complementarity induces an optimizing principal to allocate pay more equally.

Related literature

In the contract theory literature, the [Holmström \[1979\]](#) model—studying incentives for a single agent under moral hazard and imperfect observability—is a special case of our multi-agent setup. We use the first-order approach (see [Rogerson \[1985\]](#) and [Jewitt \[1988\]](#)). To our knowledge, there is not much work on how first-order conditions for contract optimality depend on spillovers.³ Indeed, the extensive literature on moral hazard beginning with [Holmström \[1982\]](#) focuses mainly on different questions. In that strand, a key question is how a principal can use observed outcomes to separately detect agents’ deviations from a desired action profile, often a nearly first-best one [see, e.g., [Mookherjee, 1984](#), [Legros and Matsushima, 1991](#), [Legros and Matthews, 1993](#)]. Several features of our model prevent such schemes.⁴ In this type of situation, when observability and fine-grained auditing of effort are fundamentally constrained, we examine how optimal contracts depend on spillovers in the production function.

Some of the closest work on optimal incentives in the presence of spillovers is found in the literature on networks. This includes, in addition to work already mentioned, papers such as [Candogan, Bimpikis, and Ozdaglar \[2012\]](#), [Bloch \[2016\]](#), [Belhaj and Deroïan \[2018\]](#), [Galeotti, Golub, and Goyal \[2020\]](#), [Gaitonde, Kleinberg, and Tardos \[2020a\]](#), [Shi \[2022\]](#), and [DeMarzo and Kaniel \[2023\]](#). Our main contribution to this literature is a study of a natural non-parametric formulation, both in terms of the production function and the form of incentives. We show that network game techniques permit some general characterizations of optimal contracts without the parametric assumptions common in the network games literature.

³[Itoh \[1991\]](#) allows for a form of spillovers in a two-agent model.

⁴In particular, the contractible outcome (which has only finitely many possible values) is stochastically determined by a one-dimensional team performance, and there is limited liability.

When we specialize to a canonical parametric environment in [Section 1.4.2](#), we also contrast the more specific implications of our analysis with existing parametric networks models—of which the closest is the contemporaneous work by [Claveria-Mayol, Milán, and Oviedo-Dávila \[2024\]](#) on optimal linear incentive contracts in quadratic network games. Both our analysis and results end up being quite different.

The problem of designing multi-agent contracts has also recently attracted attention in a new algorithmic contract theory literature—e.g., [Dütting, Ezra, Feldman, and Kesselheim \[2023\]](#), [Ezra, Feldman, and Schlesinger \[2024\]](#), and [Dütting, Ezra, Feldman, and Kesselheim \[2025\]](#). A starting point of this work is that many standard approaches to finding optimal team contracts may make heavy demands on the analyst’s knowledge of the entire production function and ability to perform computations on it. This literature studies environments with finitely many actions where combinatorial problems create obstacles to tractable optimization and examines whether contracts can be devised that achieve some fraction of optimal performance. Our approach is very different methodologically in that actions and team performance are continuous, but the results give a perspective—complementary to the algorithmic contract theory work—on parsimonious ways to assess and improve on contract performance. Recent work by [Zuo \[2024\]](#), which discusses the structure of optimization problems in a model closely related to some of our special cases, shows that there are interesting computational questions in our more continuous type of setting as well.

1.2 Model

We first present the model and then, in [Section 1.2.2](#), we comment on some issues of interpretation.

There are n agents, $N = \{1, 2, \dots, n\}$. The agents take real-valued actions $a_i \geq 0$, which can be interpreted as effort levels. These jointly determine a team *performance*, given by a function $Y : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$, which we assume is twice differentiable and strictly increasing in each of its arguments. This team performance determines the project *outcome*, an element of the finite set $\mathcal{S}$. The probability of the outcome s is $P_s(Y)$, where for any $s \in \mathcal{S}$, the function $P_s(\cdot)$ is strictly positive and twice differentiable.⁵

⁵The assumption that a probability of outcome function is strictly positive is not crucial to the analysis,

There is also one principal. (When we use pronouns, we use “she” for the principal and “he” for an agent.) The principal receives *revenue* v_s from the outcome s .⁶ The principal can make payments contingent on the project outcome but not on agents’ efforts or the team performance Y . The principal commits to a non-negative payment contingent on the outcome. If outcome s is realized, the principal pays $\tau_i(s)$ to agent i . The function $\tau : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}^n$ is called a *contract*.

We consider risk-averse agents⁷ and a risk-neutral principal.⁸ The utility to agent i from a monetary transfer is given by the function $u_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, which is strictly increasing, concave, and differentiable. Each agent also has a cost-of-effort function $C_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, which is strictly increasing, strictly convex, and twice differentiable in that agent’s action. The marginal cost at zero action is zero: $C'_i(0) = 0$. Given the contract τ , agent i chooses a_i to maximize his expected payoff from transfers minus his cost,

$$\mathcal{U}_i = \sum_{s \in \mathcal{S}} P_s(Y(a_i, a_{-i})) u_i(\tau_i(s)) - C_i(a_i).$$

The payoff for the principal given a contract τ and team performance Y is the expectation of expected revenue minus promised transfers:

$$\sum_{s \in \mathcal{S}} \left(v_s - \sum_i \tau_i(s) \right) P_s(Y).$$

The timing is as follows: The principal commits to a contract τ , and then agents simultaneously choose actions. Our solution concept for the game among the agents is pure strategy Nash equilibrium; in the remainder of the paper, we simply use the word *equilibrium* to refer to this solution.

but does simplify some statements.

⁶This should be interpreted as the principal’s valuation of that outcome realizing, gross of any payments she will make to the agents.

⁷We will specialize to the case where agents are risk neutral at times. Even in that case, the principal cannot simply “sell the firm” due to our limited liability assumption and the presence of multiple agents, so the problem remains interesting.

⁸The characterization of an optimal contract and its consequences can be extended to the case of a risk-averse principal.

There may be multiple equilibria under some contracts. Given a contract τ , we assume that agents play an equilibrium $\mathbf{a}^*(\tau)$ maximizing the principal's expected payoff. Under this selection, a principal's payoff under a contract is well-defined if at least one equilibrium exists. Among such contracts, a contract τ is *optimal* if no other contract $\tilde{\tau}$ gives the principal a higher payoff. Implicit in this definition is the assumption that contracts without equilibria can never be optimal.

Our analysis will not rely on the existence of an optimal contract, but the following argument shows an optimal contract exists if we impose a bit of additional structure.

Fact 1. Suppose that the space of contracts giving the principal a non-negative payoff is compact.⁹ Then an optimal contract exists.

As examples, this holds if the outcome is binary or if the outcome probabilities $P_s(Y)$ are uniformly bounded away from zero. The proof uses a compactness argument, taking a sequence of contracts whose payoffs converge to the supremum of attainable principal payoffs, along with their corresponding equilibria. By compactness of the contract space and action space, we can extract convergent subsequences of both the contracts and equilibrium actions. The limit contract achieves the supremum payoff because equilibria are upper-hemicontinuous in the contract and the principal's payoff function is continuous in both contracts and actions.

1.2.1 Simple success-or-failure environments

We introduce a class of *simple success-or-failure environments* that will be helpful for illustrating our results.

There are two possible outcomes $s \in \{0, 1\}$. The revenues from these outcomes are normalized so that $v_1 = 1$ and $v_0 = 0$. These can be interpreted as success or failure of the project. The probability of success is $P(Y)$, where the function $P(\cdot)$ is strictly increasing and twice differentiable.

Each agent has a quadratic cost of effort.¹⁰ Agent i maximizes the expected payoff given by

⁹We exclude any contracts where no equilibrium exists from this space.

¹⁰This is not substantively restrictive in that, under smoothness assumptions, a monotone transformation

the expression:

$$\mathcal{U}_i = P(Y)u_i(\tau_i(1)) + (1 - P(Y))u_i(\tau_i(0)) - \frac{a_i^2}{2}.$$

Fact 2. In simple success-or-failure environments, it is optimal for the principal to pay nothing at the failure outcome—that is, $\tau_i(0) = 0$ for all agents i . A contract can then be represented by a n -dimensional vector $\boldsymbol{\tau} \in \mathbb{R}_{\geq 0}^n$ consisting of payments in the success outcome.

The reason is simple: agents' incentives depend only on the difference $u_i(\tau_i(1)) - u_i(\tau_i(0))$ between utility conditional on success and failure, so we can shift payments and assume $\tau_i(0) = 0$ and $u_i(0) = 0$. This shift can only improve the principal's payoff, so it is without loss of optimality in the principal's problem. We can interpret $\tau_i(1)$ as an equity share in the project's output.

Linear-quadratic production functions

Within this class of environments, we can define an important leading example, which we will analyze in more detail after giving our main result.

Consider *linear-quadratic production functions* of the form

$$Y(\mathbf{a}) = \sum_{i=1}^n k_i a_i + \frac{1}{2} \sum_{i,j=1}^n G_{ij} a_i a_j, \quad (1.1)$$

where k_i captures agent i 's standalone productivity and the entrywise nonnegative symmetric matrix $\mathbf{G}$ is the technological complementarity matrix. Its off-diagonal entries capture bilateral complementarities between agents' efforts, while G_{ii} captures curvature in agent i 's own action.¹¹

Fix a concave success probability function $P(\cdot)$. In the appendix, [Lemma 6](#) shows that a fixed contract $\boldsymbol{\tau}$ induces a unique Nash equilibrium. Under those conditions, the equilibrium

can be applied to a_i to achieve this form.

¹¹This functional form is a second-order approximation to any smooth production function Y , with $\mathbf{G}$ being the Hessian of Y with respect to actions at equilibrium.

actions $\mathbf{a}^*$ and team performance $Y^* = Y(\mathbf{a}^*)$ solve

$$[\mathbf{I} - P'(Y^*)\mathbf{T}\mathbf{G}]\mathbf{a}^* = P'(Y^*)\mathbf{T}\mathbf{k}, \quad (1.2)$$

where $\mathbf{T}$ is a diagonal matrix with entries $T_{ii} = u_i(\tau_i)$. The equilibrium structure is reminiscent of standard models of network games with linear best responses (Ballester et al. [2006]), but two nuances complicate the situation: non-linear success probability $P(Y)$, as well as the appearance of the endogenous object $\mathbf{T}$ as part of the matrix $P'(Y^*)\mathbf{T}\mathbf{G}$ that captures incentive spillovers.

An instructive special case arises when the symmetric matrix $\mathbf{G}$ has rank one: $\mathbf{G} = \rho\boldsymbol{\beta}\boldsymbol{\beta}^\top$ for some nonnegative vector $\boldsymbol{\beta}$ and scalar $\rho > 0$. In this case, the production function can be written as

$$Y(\mathbf{a}) = \sum_{i=1}^n k_i a_i + \frac{\rho}{2} \left(\sum_{i=1}^n \beta_i a_i \right)^2, \quad (1.3)$$

where β_i can be interpreted as the degree to which agent i 's effort complements others', and ρ governs the overall strength of complementarities.

Both the case with general $\mathbf{G}$ and the rank-one case will be important for applications.

1.2.2 Remarks on the model

The team performance Y is real-valued, but the outcome s is discrete. This assumption need not substantively restrict the scope of the model since the outcome can be, for example, a revenue rounded to the nearest cent. On the other hand, the fact that outcomes are mediated by a one-dimensional performance level is important. We discuss in the paper's concluding remarks the issue of extending the analysis to outcomes determined by a higher-dimensional function of efforts.

We assume that the firm's output is the only contractible consequence of any agent's effort. In other words, agents cannot be paid directly for their efforts a_i . In this we follow the literature stemming from Holmström [1982], which is motivated by the practical features of contracts and the fact that a_i and Y are abstractions that may not have any measurable real-world counterpart, among other considerations.

We do not impose restrictions on the space of contracts beyond non-negativity of payments. (In particular, we allow the principal to lose money at some outcomes with the hope of inducing a higher action.) In practice, compensation may be restricted to particular simpler classes of contracts. For example, equity contracts award an agent the same share of profit in every state. In many such settings, analogues of our balance condition can be derived, as we demonstrate for the case of equity contracts in [Section A.5](#).

We do not assume that equilibria exist under all contracts in the formulation of the model or our analysis. For the contract that pays zero in all states, there is always an equilibrium in which agents take zero actions; this ensures that the principal’s value is well-defined. For other contracts, existence needs to be analyzed in the environment of interest; for example, [Lemma 6](#) establishes existence and uniqueness in the parametric model of [Section 1.2.1](#).

1.3 Optimal Contracts

This section presents and interprets the characterization of optimal contracts, beginning by defining its ingredients.

1.3.1 Key objects

We start with some notation. Fix a contract τ and a corresponding principal-optimal equilibrium $\mathbf{a}^*(\tau)$. Let Y^* be the team performance under this equilibrium.

Definition (Marginal productivity). *Let $\nabla Y(\mathbf{a}^*)$ be the gradient of $Y(\cdot)$ at $\mathbf{a}^*$, restricted to the agents that take a strictly positive action. We define the marginal productivity vector α as*

$$\alpha := \nabla Y(\mathbf{a}^*).$$

The entry α_i captures the marginal effect of i ’s action on team performance. We use *productivity* as shorthand for marginal productivity. Here, as in other cases, the quantities in the expression for α are evaluated at the equilibrium $\mathbf{a}^*$, under a certain contract τ —but we often omit this dependence on the equilibrium and the contract notationally for brevity.

We turn next to centrality, which concerns how incentives propagate through the team. Let $\mathbf{G}$ denote the *technological complementarity matrix*: the Hessian matrix of Y with respect to agent actions, restricted to agents that take a strictly positive action in $\mathbf{a}^*$. Formally, for agents j and k such that $a_j^* > 0$ and $a_k^* > 0$, define

$$G_{jk} := \frac{\partial^2 Y}{\partial a_k \partial a_j}.$$

Let the *marginal payment utility* matrix $\mathbf{U}$ be a diagonal matrix where

$$U_{jj} := \sum_{s \in \mathcal{S}} P'_s(Y^*) u_j(\tau_j(s))$$

is the marginal change in agent j 's utility from payments when team performance increases. The increase in Y changes all the probabilities of outcomes, and the agent's utility from these outcomes is given by $u_j(\tau_j(s))$, where the contract is held fixed.

Definition (Spillover matrix and centrality). *Let $\mathbf{H}$ be the diagonal matrix with $H_{jj} := C''_j(a_j^*)$. Whenever $\mathbf{H}$ is nonsingular, define the spillover matrix*

$$\mathbf{S} := \mathbf{H}^{-1} \mathbf{U} \mathbf{G}.$$

Whenever $\mathbf{I} - \mathbf{S}$ is nonsingular, the centrality vector is

$$\mathbf{c}^T := \boldsymbol{\alpha}^T [\mathbf{I} - \mathbf{S}]^{-1}. \quad (1.4)$$

The entry c_i can be thought of as the total effect on team performance induced by a marginal change in agent i 's incentive to increase a_i . This effect is inclusive of spillovers on others' incentives through strategic interactions. To explain the role of $\mathbf{H}$: when we analyze how equilibrium actions vary with contract perturbations, an agent's best response is less sensitive to incentives when $C''_j(a_j^*)$ is larger, and $\mathbf{H}$ allows us to capture this effect. Centrality is well-defined when the inverse in the formula exists, which will be ensured under the conditions of our results.

Productivity and centrality take simpler forms in simple success-or-failure environments. In

that case, the matrix $\mathbf{H}$ is equal to the identity matrix $\mathbf{I}$ because the cost of each agent's action is $\frac{1}{2}a_i^2$. So

$$\alpha_i = \frac{\partial Y}{\partial a_i} \quad \text{and} \quad \mathbf{c}^\top = \boldsymbol{\alpha}^\top [\mathbf{I} - P'(Y)\mathbf{T}\mathbf{G}]^{-1}, \quad (1.5)$$

where $\mathbf{T} = \text{diag}(u_1(\tau_1), \dots, u_n(\tau_n))$. If we further specialize to the rank-one setting of [Section 1.2.1](#), then the technological complementarity matrix is $\mathbf{G} = \rho\boldsymbol{\beta}\boldsymbol{\beta}^\top$ and productivities are

$$\alpha_i = \frac{\partial Y}{\partial a_i} = k_i + \rho\beta_i x^*,$$

where

$$x^* := \boldsymbol{\beta}^\top \mathbf{a}^* = \sum_{j=1}^n \beta_j a_j^*$$

is the complementarity-weighted sum of equilibrium actions.

1.3.2 Balance condition across agents

In this section, we present our main result: a balance condition across agents at each outcome realization.

We will state a necessary condition for an optimal contract $\boldsymbol{\tau}^*$ under the following assumption, which will be maintained from now on.

Assumption 1. *A differentiable selection $\mathbf{a}^*(\boldsymbol{\tau})$ from the equilibrium correspondence can be defined in a neighborhood of $\boldsymbol{\tau}^*$.*

This assumption stipulates that equilibrium varies differentiably as we slightly perturb the contract in a neighborhood of the optimum. [Section A.2](#) gives sufficient conditions for the applicability of the first-order approach, including explicit ones on the primitives that extend those familiar from single agent problems [[Rogerson, 1985](#)]. (See also [Section 1.4](#) for specific production functions $Y(\mathbf{a})$ under which the first-order approach applies.) The one-dimensional aggregator $Y(\mathbf{a})$ plays a crucial role here. Whereas in general providing sufficient conditions for the first-order approach in arbitrary multi-agent settings is quite challenging, in our framework it is much more tractable.

The following result characterizes optimal contracts.

Theorem 1. *Suppose τ^* is an optimal contract and Y^* is the induced team performance. There exist constants λ_s such that for any agent i receiving a positive payment under an outcome s , we have*

$$\alpha_i c_i \cdot \frac{u'_i(\tau_i^*(s))}{C'''_i(a_i^*)} = \lambda_s. \quad (\text{BC})$$

Moreover, the outcome-dependent constants λ_s satisfy $\lambda_s \propto \frac{P_s(Y^*)}{P'_s(Y^*)}$.

This result says that optimal incentives require balance to hold, with the product on the left being equal across agents. More informally, the balance condition (BC) states that under an outcome s ,

$$\text{productivity}_i \cdot \text{centrality}_i \cdot \text{responsiveness}_i = \text{constant}$$

for all agents paid when that outcome occurs. Below, we will give more intuition for why this is a necessary condition.

In simple success-or-failure environments with all agents having utility function $u_i(\tau) = \tau$, we obtain a simpler statement: there exists a constant λ such that

$$\alpha_i c_i = \lambda$$

for all agents i receiving positive payments conditional on success. Recall that in this case α_i and c_i take the simpler forms given in (1.5).

We note that the proof does not rely on the induced team performance Y^* being optimal. The balance condition at the optimal contract would hold if the principal instead wanted to implement any desired level of performance with minimal (expected) transfers to agents. Solving for the minimal payments implementing a given outcome distribution and then optimizing over outcome distributions is a standard approach in single-agent settings (e.g., Grossman and Hart [1983]). With multiple agents, the first step also involves allocating incentives across agents.

The key to the proof, formalized in the following lemma, is calculating the effect on team performance of increasing an agent's payment under a given outcome. Assumption 1 ensures

that these perturbations are well-defined.

Lemma 1. *Suppose τ^* is an optimal contract with corresponding equilibrium actions $\mathbf{a}^*$ and team performance Y^* . Consider any agent i receiving a positive payment at some outcome. For any outcome s , the derivative of team performance in $\tau_i(s)$, evaluated at τ^* , is*

$$\frac{dY}{d\tau_i(s)} = lP'_s(Y^*)\alpha_i c_i \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)},$$

where l is independent of i and s .

A complete proof for the result above is provided in [Section A.1](#). We provide some intuition for the various terms in the expression in the lemma.

Intuition for the proof: The lemma characterizes the effect on team performance of increasing the transfer to agent i under outcome s . We can decompose this effect as the product of:

- (i) a factor $P'_s(Y^*)\alpha_i$ capturing the sensitivity of the probability of the outcome s to i 's effort;
- (ii) an action-responsiveness factor $\frac{u'_i(\tau_i(s))}{C''_i(a_i^*)}$: the numerator governs how the transfer perturbation changes i 's marginal incentive, and the denominator converts that marginal-incentive change into a direct action response;
- (iii) a term c_i capturing the spillovers from changing i 's incentive to exert effort;
- (iv) the constant l , which depends on the curvature of the probability $P_s(Y)$.

We focus on the first three factors and defer treatment of the fourth term, which is not central in the basic intuition, to the formal proof in the appendix.

The first two factors, (i) and (ii), measure the principal's ability to directly induce agent i to change his action by rewarding him when outcome s is realized. The change in agent i 's marginal payoff from effort as $\tau_i(s)$ increases is $P'_s(Y^*)\frac{\partial Y}{\partial a_i}u'_i(\tau_i(s))$. Dividing by $C''_i(a_i^*)$ converts this marginal-incentive change into the direct change in i 's optimal action. An agent is easier to incentivize when his marginal utility of money is higher and when his marginal

cost of effort increases less steeply.

Multiplying by term (iii), the centrality c_i , translates from this direct effect on i 's action to the overall change in equilibrium team performance. This effect is at the heart of our use of network theory; a detailed intuition for this appears in [Section 1.3.2](#) just below.

First, though, we sketch why [Lemma 1](#) implies [Theorem 1](#). (A formal proof is provided in [Section A.1](#).) We want to show that the balance condition

$$\alpha_i c_i \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)} = \alpha_j c_j \cdot \frac{u'_j(\tau_j^*(s))}{C''_j(a_j^*)},$$

must hold under an optimal contract. Suppose that the principal would benefit from a slightly higher team performance (the case in which the principal prefers a slightly lower team performance proceeds analogously). [Lemma 1](#) shows that the change in team performance from increasing agent i 's payment under outcome s is equal to $\alpha_i c_i \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)}$ times terms independent of i , and similarly for agent j . If we had

$$\alpha_i c_i \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)} > \alpha_j c_j \cdot \frac{u'_j(\tau_j^*(s))}{C''_j(a_j^*)},$$

it would be profitable for the principal to pay agent i slightly more and agent j slightly less under outcome s . The same argument holds in the opposite direction, so the balance condition is necessary for the contract to be optimal.

What centrality captures

This section gives an intuition for the role of network centrality in our characterization, and the details of how network centrality is applied in our setting. For simplicity, we focus on the simple success-or-failure case, with $\mathbf{H} = \mathbf{I}$ (which is essentially a normalization), and recall $\mathbf{S} = P'(Y)\mathbf{T}\mathbf{G}$ is the spillover matrix induced by the complementarity matrix $\mathbf{G}$.

When the spectral radius of $\mathbf{S}$ is less than 1, we can write

$$\mathbf{c}^\top = \boldsymbol{\alpha}^\top \left[\sum_{\ell=0}^{\infty} \mathbf{S}^\ell \right].$$

Economically, the powers capture the effects of the initial increase in i 's action ($\ell = 0$), the resulting changes in each agent's best response ($\ell = 1$), the further changes in best responses induced by these, etc. The entries of $\mathbf{S}$ describe how j 's increased effort spills over to affect i 's marginal incentives, accounting for technological complementarities and contracts. The form of spillovers makes sense: the direct dependence of i 's best response on j 's action is proportional to $P'(Y)$ —the sensitivity of success probability to team performance Y ; $u_i(\tau_i)$ —agent i 's utility conditional on success; and to the complementarity of i with j . Unlike in standard models of network games with linear best-responses, the spillover matrix $\mathbf{S}$, and therefore the resulting centrality measure, depend endogenously on the contract and equilibrium.

In terms of graph statistics, entry (i, j) of $\mathbf{S}^\ell$ counts walks of length ℓ from j to i in the network of spillovers, with each step in the walk from a node j' to a node i' contributing a factor $S_{i'j'}$ to the weight of a walk.¹² To compute centralities, the entries of these powers are then multiplied by productivities α_i : an agent j is more central if the ripple effects of motivating j increase the incentives of more productive agents.

Further comments on the result and the modeling behind it.

As our discussion makes clear, [Theorem 1](#) essentially follows from a decomposition of the terms in the first-order conditions for contract optimality. One contribution of our work, as we have already mentioned, is formulating a framework in which the first-order approach is valid under conditions that extend those justifying the first-order approach from single-agent contracts ([Section A.2](#)).

Another contribution is formulating the framework so that these conditions coming from the first-order approach are as simple and intuitive as possible. In particular, if we simply differentiate i 's incentives in j 's actions, second derivatives of $P(Y)$ show up. But such terms do not appear in our definition of centrality. In the proof of our main result, we show that changing the second derivative $P''(Y^*)$ rescales all relevant expressions involving the spillover matrix $\mathbf{S}$ by a common factor, and so second derivative terms can be ignored when checking whether redistributing incentives between agents is profitable. This simplification

¹²See [Ballester et al. \[2006\]](#) and [Bloch, Jackson, and Tebaldi \[2023\]](#) for more on these centrality measures.

relies crucially on the model feature that agents' actions contribute to production via one-dimensional team performance Y .

In the remaining sections of the paper, we build on the insights from the implicit characterization and derive explicit connections between economic primitives and optimal contracts when the production function takes various functional forms.

1.3.3 Comparisons across agents and outcomes.

Before turning to these more applied consequences of [Theorem 1](#), we give several basic corollaries describing how incentives are distributed across agents and across outcomes. We first give conditions implying an ordinal ranking of payments to agents that does not depend on the particular outcome. We then provide a multi-agent version of a result from [Holmström \[1979\]](#) on the optimal allocation of payments across outcomes rather than agents.

Ranking agents.

Agents can be ranked in terms of payments at the optimal contract. To see this, we establish a relationship between the marginal utilities of agents. An implication of [Theorem 1](#) is that the ratio between any two agents' marginal utilities is the same at every outcome for which both receive positive transfers.

Corollary 1. *Consider an optimal contract τ^* . Let $\mathcal{S}_{ij}^*$ be the set of outcomes at which agents i and j both receive a positive payment. For any outcome $s \in \mathcal{S}_{ij}^*$, we have*

$$\frac{u'_i(\tau_i^*(s))}{u'_j(\tau_j^*(s))} = \frac{\alpha_j c_j}{\alpha_i c_i} \cdot \frac{C'_i(a_i^*)}{C'_j(a_j^*)}.$$

Intuitively, since outcome probabilities are determined by a joint team performance, agents' incentives should vary across outcomes in similar ways. The corollary formalizes this intuition in terms of marginal utilities in each outcome.¹³

¹³This contrasts with a literature on optimal compensation when the observed outcome can be used to identify individuals who deviated from a desired level of effort (e.g., [Holmström \[1982\]](#) and [Legros and Matthews \[1993\]](#)).

The corollary only applies when the set of outcomes at which agents i and j both receive a payment is non-empty. Determining when an agent is paid at a given outcome can be complicated in general, but it is easy to construct settings where the corollary applies. In [Section A.4](#), for example, we give a class of environments in which an Inada condition guarantees that all agents are paid at all outcomes where $P'_s(Y^*) > 0$ (and no other outcomes).

When agents have identical utility functions, agents can be ranked so that an optimal contract provides stronger incentives to more highly ranked agents.

Proposition 1. *Suppose that τ^* is an optimal contract. If a pair of agents i and j have identical strictly concave utility functions $u_i(\cdot) = u_j(\cdot)$, then*

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S} \text{ or } \tau_j^*(s) \geq \tau_i^*(s) \text{ for all } s \in \mathcal{S}$$

(or both).

The intuition is simple: for two agents that derive the same value from a monetary transfer, the agent with a greater overall effect on team performance at the optimal contract must be receiving a higher payment.

When all agents have an identical utility function, the optimal contract induces a complete ranking on the agents. The relative magnitude of payments across agents depends on the environment. This becomes evident in the parametric example discussed further in [Section 1.4.2](#).

Allocation across outcomes.

A second implication of the main balance result is a relationship between a single agent's marginal utility across outcomes.

Corollary 2. *Suppose τ^* is an optimal contract and Y^* is the induced team performance. If agent i receives positive payments under outcomes s_1 and s_2 , then*

$$\frac{u'_i(\tau_i^*(s_1))}{u'_i(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P_{s_1}(Y^*)} \cdot \frac{P'_{s_2}(Y^*)}{P_{s_2}(Y^*)}.$$

The corollary states that the marginal utility under each outcome is proportional to the probability of that outcome divided by the marginal change in that probability as team performance increases. That is, agents are paid more in outcomes that are less likely and more responsive to team performance. This result generalizes a result in the single-agent setting of [Holmström \[1979\]](#) concerning how a single agent’s payments should be allocated across outcomes.

A straightforward application of [Corollary 2](#) characterizes the set of outcomes at which an agent receives a positive payment. If an agent receives a positive payment at some outcome, the outcomes at which he receives a positive payment must all either have a positive marginal probability at equilibrium team performance, or a negative marginal probability. When the team performance function $Y(\cdot)$ is strictly increasing in each of its arguments, the outcomes at which an agent receives a positive payment all have a positive marginal probability at equilibrium team performance. (This is formalized as [Lemma 4](#) in the Appendix).

In the special case that an agent is risk-neutral, the corollary has a stronger implication: agents are only paid under the outcome(s) maximizing $P'_s(Y^*)/P_s(Y^*)$. It is straightforward to construct functions $P_s(Y)$ where there can only be one such outcome. Risk neutrality removes the principal’s motives to diversify payments across outcomes, so compensation can be concentrated in the outcome providing the most incentive power.

1.4 Implications: Whom to incentivize in teams?

The balance condition in [Theorem 1](#) characterizes optimal contracts implicitly. We now derive explicit solutions and comparative statics in two tractable settings that illuminate how exogenous parameters determine the balance condition and the resulting contracts.

First, we study a setting in which agents differ in their intrinsic capacities for productivity and complementarity. A single complementarity parameter governs network interactions with all agents, yielding a rank-one network. This exercise allows us to examine which of these exogenous traits should primarily drive compensation.

A second application focuses on better understanding centrality in general networks. By

turning off differences in standalone productivity, we can explicitly characterize how a heterogeneous technological complementarity matrix shapes agents' centralities at the optimal contract. We find some interesting non-monotonicities that sharply distinguish our theory from standard network models. This illustrates both the importance and the subtlety of the fact that centralities are evaluated at endogenously determined responses to optimal contracts, rather than in an exogenous network.

1.4.1 Rank-one networks: Standalone productivity versus complementarity

We return to the rank-one, linear-quadratic setting of [Section 1.2.1](#). This setting allows heterogeneity in both standalone productivities k_i and complementarity parameters β_i , while maintaining tractability through the rank-one structure of the Hessian $\mathbf{G} = \rho\boldsymbol{\beta}\boldsymbol{\beta}^\top$. Throughout this section, we maintain the assumption that the parameters satisfy the bounds $k_i \in [\underline{k}, \bar{k}]$ for some $\bar{k} > \underline{k} > 0$ and $\beta_i \in [\underline{\beta}, \bar{\beta}]$ for some $\bar{\beta} > \underline{\beta} > 0$.

Characterization of optimal contracts

Consider any contract $\boldsymbol{\tau}$ with corresponding equilibrium actions $\mathbf{a}^*$ and team performance Y^* . Define the *spillover matrix*

$$\mathbf{S} := \rho P'(Y^*) \mathbf{T} \boldsymbol{\beta} \boldsymbol{\beta}^\top,$$

where $\mathbf{T} = \text{diag}(u_1(\tau_1), \dots, u_n(\tau_n))$ collects utilities from the success payments (with $\tau_i = \tau_i(1)$, recalling [Fact 2](#)). The spectral radius $\mu(\mathbf{S})$ of this matrix determines whether standalone productivity or complementarity governs optimal pay.

A key result, which follows from our general balance condition, characterizes optimal contracts in terms of a quadratic form in the primitive parameters.

Lemma 2. *Consider any optimal contract $\boldsymbol{\tau}^*$ in the rank-one setting of [Section 1.2.1](#). There exist constants $B_1 \geq 0$ and $B_2 \geq 0$ such that for all agents i receiving positive pay,*

$$u'_i(\tau_i^*) \propto (k_i^2 + B_1 k_i \beta_i + B_2 \beta_i^2)^{-1}.$$

This lemma shows that optimal pay depends on a quadratic form in the agent's standalone productivity k_i and complementarity β_i . The coefficients B_1 and B_2 depend on the equilibrium and determine the relative importance of standalone productivity versus complementarity in setting compensation.

Spillover regimes

The following theorem establishes that the strength of spillovers determines which characteristic—standalone productivity or complementarity—dominates optimal pay.

Theorem 2. *In the rank-one setting of [Section 1.2.1](#), fix all primitives other than the complementarity parameter ρ . Consider a sequence of complementarity parameters $\rho^{(m)}$ with corresponding optimal contracts $\tau^{(m)}$ and induced spillover matrices $\mathbf{S}^{(m)}$. Write $\mu^{(m)}$ for the spectral radius of $\mathbf{S}^{(m)}$. For any pair of agents i and j that receive positive pay for all sufficiently large m ,*

$$(i) \text{ If } \mu^{(m)} \rightarrow 0, \text{ then } \lim_{m \rightarrow \infty} \frac{u'_i(\tau_i^{(m)})}{u'_j(\tau_j^{(m)})} = \frac{k_j^2}{k_i^2}.$$

$$(ii) \text{ If } \mu^{(m)} \rightarrow 1, \text{ then } \lim_{m \rightarrow \infty} \frac{u'_i(\tau_i^{(m)})}{u'_j(\tau_j^{(m)})} = \frac{\beta_j^2}{\beta_i^2}.$$

The theorem establishes a sharp dichotomy. When spillovers are weak (low μ), the propagation of incentives across agents is limited, so pay primarily reflects direct productivity k_i . When spillovers are strong (high μ), the network of complementarities dominates, and an agent's position in the interaction structure—captured by β_i —becomes the primary determinant of compensation. We note that the spectral radius $\mu^{(m)}$ varies endogenously with the complementarity parameter $\rho^{(m)}$, fixing the rest of the environment.

Beyond delineating these regimes, [Theorem 2](#) reveals that heterogeneity in productivity or complementarity is, in a sense, *magnified* under optimal contracts: the ratio of marginal utilities depends on the *square* of the ratio of the relevant parameters. This is a consequence (a non-obvious one, in our view) of the specific type of incentive problem we have here, where the proceeds of a single output must be shared out to compensate the agents.

The characterization in [Lemma 2](#) suggests a regression specification for studying compensation in teams. If k_i and β_i can be measured (or proxied), then optimal pay should depend on a quadratic form in these variables. [Theorem 2](#) further predicts that the relative weights should vary systematically with the strength of complementarities: in industries or teams with stronger spillovers, the β terms should receive more weight.

We provide a numerical example to illustrate how the optimal contract's behavior depends on the strength of spillovers, as described above. This effect is particularly relevant in environments where the team includes members with high standalone productivities and weak complementarities.

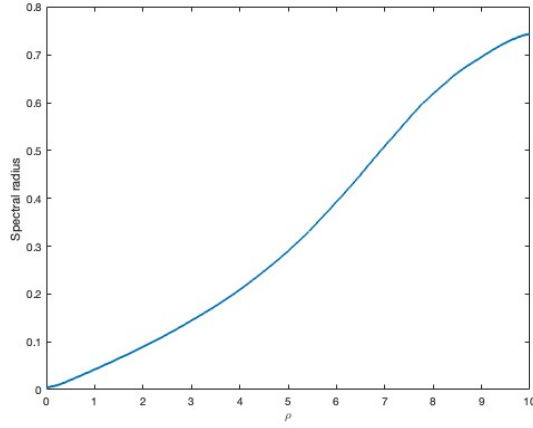
We define a team of four agents, partitioned evenly into two groups $\mathcal{T}_1$ and $\mathcal{T}_2$. Agents in group $\mathcal{T}_1$ have standalone productivity $k = 0.3$ and complementarity $\beta = 0.15$, whereas agents in group $\mathcal{T}_2$ have standalone productivity $k = 0.15$ and complementarity $\beta = 0.3$. So the first group has higher standalone productivity and weaker complementarity. The probability of success is an exponential function:

$$P(Y) = 1 - e^{-Y} \quad \text{for all } Y \geq 0.$$

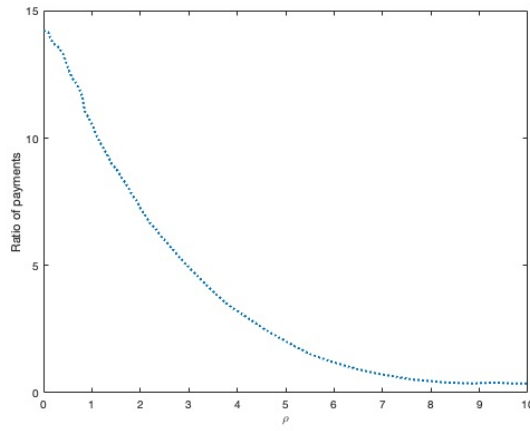
Each agent obtains utility $u_i(\tau_i) = \sqrt{\tau_i}$ from a transfer τ_i .

[Figure 1.1\(a\)](#) plots the spectral radius of the spillover matrix induced by the optimal contract as ρ varies. By symmetry, the two agents in a group $\mathcal{T}_i$ receive identical pay under an optimal contract. [Figure 1.1\(b\)](#) illustrates how the ratio of payments to agents in $\mathcal{T}_1$ relative to those in $\mathcal{T}_2$ varies with ρ .

As depicted in [Figure 1.1\(b\)](#), when ρ is close to 0, cross-sectional variation in pay is driven primarily by differences in standalone productivities: agents in $\mathcal{T}_1$ receive approximately sixteen times the payment of agents in $\mathcal{T}_2$. For intermediate values of ρ , payments at the optimal contract are approximately equal across agents. For large values of ρ , it is optimal to pay agents in $\mathcal{T}_2$ much more than agents in $\mathcal{T}_1$.



(a) Spectral radius of the spillover matrix



(b) Ratio of payments

Figure 1.1: Panel (a) displays the spectral radius of the spillover matrix as ρ varies over the range $[0, 10]$. Panel (b) reports the ratio of payments to agents in group $\mathcal{T}_1$ relative to those in group $\mathcal{T}_2$ at the optimal contract as a function of ρ .

The importance of spillovers in contract design

The preceding analysis describes how the structure of an optimal contract changes with spillovers. To underscore the importance of accounting for spillovers, we now compare the principal's profits under the optimal contract to profits under a misspecified contract neglecting spillovers.

Formally, the misspecified contract is the optimal contract if $\rho = 0$ (and all other parameters are unchanged). This misspecified contract equalizes $k_i^2 u'_i(\tau_i^h)$ across agents. In contrast, as established in [Lemma 2](#), the correctly-specified optimal contract τ^* equalizes

$$(k_i^2 + B_1 k_i \beta_i + B_2 \beta_i^2) u'_i(\tau_i^*)$$

across agents for some constants $B_1 > 0$ and $B_2 > 0$. Comparing profits across these contracts therefore isolates the cost to the principal of ignoring spillover effects.

We perform this comparison in the setting of the numerical example in the preceding section with global complementarity parameter $\rho = 3$. We modify the probability of success to be an exponential function

$$P(Y) = 1 - e^{-\xi Y} \quad \text{for all } Y \geq 0$$

with a flexible parameter $\xi \geq 0$ (which was set to 1 in the previous example).

[Figure 1.2](#) depicts the ratio of profits at an optimal contract τ^* to profits at the benchmark contract τ^h . The optimal contract significantly outperforms the benchmark as long as failures are not too rare. The gap grows very large when the probability of success becomes a steeper function of team performance (large ξ).

1.4.2 General networks

The rank-one setting in [Section 1.4.1](#) allowed rich individual heterogeneity but restricted the complementarity structure. Ideally, we would have a similar level of insight into how the endogenous moving parts of our balance condition depend on primitives with arbitrary network structures $\mathbf{G}$. Unfortunately, we do not know how to obtain explicit characterizations in the linear-quadratic setting with general networks and heterogeneous standalone productivities.

Considerable insight can be gained, however, from a natural special case. Staying within the linear-quadratic success-or-failure setting of [Section 1.2.1](#), we make all standalone productivities equal and assume risk-neutrality, i.e. $u_i(\tau_i) = \tau_i$, for all agents. This setting, with an arbitrary $\mathbf{G}$, will let us get some traction on how network structure alone shapes optimal

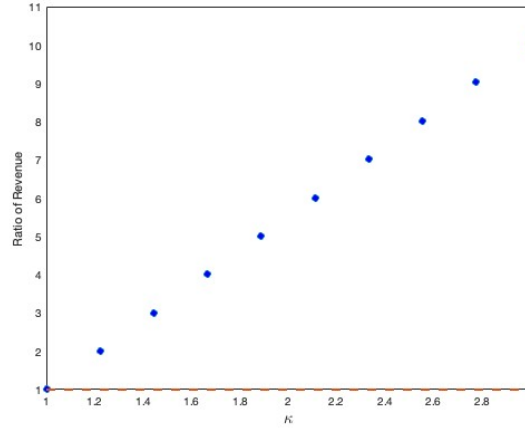


Figure 1.2: The ratio of profits at an optimal contract and profits at a benchmark contract which ignores spillovers. The red dashed line indicates a ratio of 1.

contracts.

Consider any contract τ with corresponding equilibrium actions $\mathbf{a}^*$. At equilibrium, an agent i chooses a positive action if $\tau_i > 0$ and an action of zero otherwise. An agent is said to be *active* under a given contract τ if he receives a positive payment $\tau_i > 0$ and *inactive* otherwise.¹⁴ We will focus on characterizing the optimal contract payments and equilibrium actions among active agents.

Proposition 2. *In the setting of this section, suppose τ^* is an optimal contract and $\mathbf{a}^*$ and Y^* are the induced equilibrium actions and team performance, respectively. The following properties are satisfied:*

- (a) *For any two active agents i and j , we have $\alpha_i = \alpha_j$ and $c_i = c_j$.*
- (b) *Balanced neighborhood actions: There is a constant $\lambda' > 0$ such that for all active agents i , we have $(\mathbf{G}\mathbf{a}^*)_i = \lambda'$.*
- (c) *Balanced neighborhood equity: There is a constant $\lambda > 0$ such that for all active agents i , we have $(\mathbf{G}\tau^*)_i = \lambda$.*

¹⁴The extensive margin leads to a rich set of questions—which agents are active at an optimal contract. We leave these questions for future work.

The first part of the result states that all active agents have equal productivities and equal centralities. This is much stronger than what the balance condition guarantees, which is that $\alpha_i c_i$ is constant across active agents i .

The property of balanced neighborhood actions states that for each active agent i , the sum of actions of neighbors of i , weighted by the strength of i 's connections to those neighbors in $\mathbf{G}$, is equal to the same number, λ' . Similarly, the property of balanced neighborhood equity says that for each active agent i , the sum $\sum_j G_{ij} \tau_j$ of shares given to neighbors of i , weighted by the strength of i 's connections to those neighbors in $\mathbf{G}$, is equal to the same number (i.e., is not dependent on i).

Why must neighborhood action balance hold? Intuitively, if balance failed and an agent i had strong links and neighbors taking high actions, it would be profitable to reallocate incentives from other agents to i . (This relies on equal standalone productivities and did not need to hold in [Section 1.4.1](#).) One implication of this balance is that differences in the exogenous complementarity structure are attenuated by the principal's optimization. If an agent i has stronger links G_{ij} , then to satisfy balanced neighborhood actions, i.e. holding $\sum_j G_{ij} a_j$ constant, i 's neighbors must take lower actions. This “balancing” force diminishes the complementarities that i experiences. So, while the example here has a lot of structure—and, in particular, the equality of standalone productivity plays an important role—it cleanly makes the point that comparative statics as we vary “complementarity parameters” in general networks are subtle.

In fact, the structure of this setting permits an explicit solution for optimal equity shares and the associated equilibrium actions, which allows us to explore these subtleties more precisely. Assume the relevant adjacency matrix $\mathbf{G}$ is invertible, which holds for generic weighted networks. At an optimal solution, the payments and actions of active agent i satisfy

$$\tau_i^* \propto \left(\tilde{\mathbf{G}}^{-1} \mathbf{1} \right)_i,$$

where $\tilde{\mathbf{G}}$ is the subnetwork of active agents for that payment allocation. This follows immediately from the balance conditions in [Proposition 2](#). Thus, the sum of row i in $\tilde{\mathbf{G}}^{-1}$ is an index characterizing i 's compensation and equilibrium action. Thanks to this explicit description, we can see that these outcomes behave quite differently from standard mea-

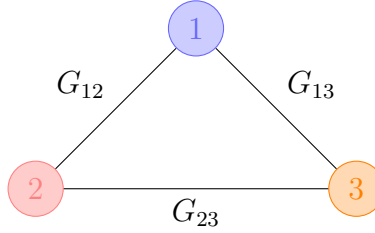


Figure 1.3: Three agent weighted graph with weights G_{12} , G_{13} , and G_{23} .

asures of centrality such as Katz–Bonacich centrality. In particular, unlike the case with such standard measures, increasing the strength of an agent’s links in $\mathbf{G}$ need not increase the compensation index, as we illustrate in the next section.

Comparative statics

In this section, we explore how the optimal contract, as well as the agents’ and principal’s payoffs, vary with the technology of production. The simple form of the team performance function Y in our environment, as well as the explicit characterization of incentives and outcomes, facilitate this exercise. The results demonstrate some interesting tensions between the principal’s and the agents’ interests.

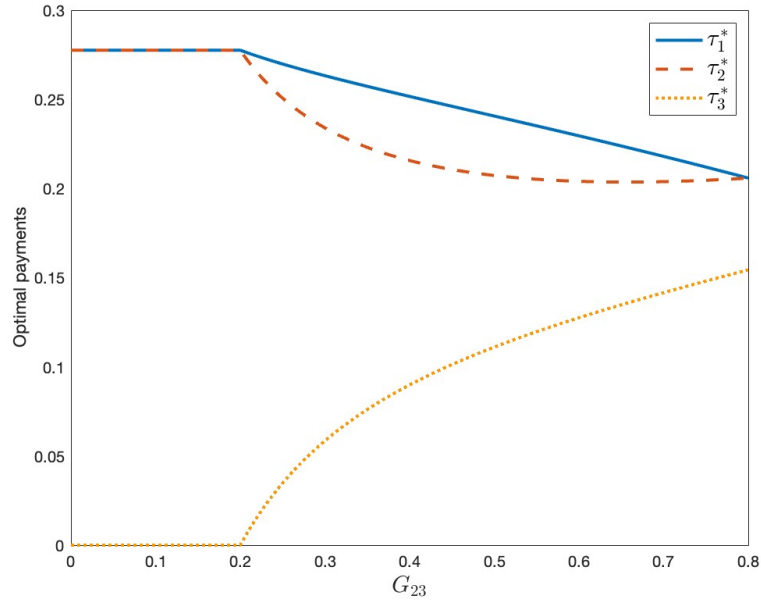
We look at how the principal’s and agents’ payoffs vary as the network changes.

Proposition 3. *The principal’s payoff is weakly increasing in the edge weight $G_{ij} = G_{ji}$.*

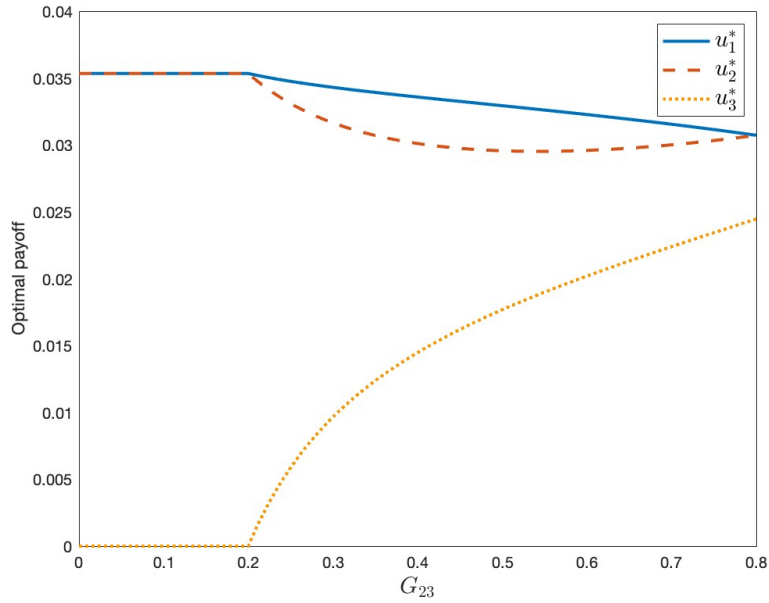
The principal obtains weakly higher profits from an increase in edge weights. However, it need not be the case that agents prefer such a perturbation. We will illustrate this through a network on three agents (see [Figure 1.3](#)).

Without loss of generality, we can assume $G_{12} \geq G_{13} \geq G_{23}$ and choose the normalization $G_{12} = 1$, so that the adjacency matrix is

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & G_{13} \\ 1 & 0 & G_{23} \\ G_{13} & G_{23} & 0 \end{bmatrix}.$$



(a) Optimal payments



(b) Payoffs under optimal payments

Figure 1.4: The optimal payments and resulting equilibrium payoffs as a function of the weight G_{23} . Here $G_{13} = 0.8$ while $P(Y) = \min\{0.5Y, 1\}$ (this could be replaced by a smooth, strictly concave P without affecting the results.). In both diagrams, the curve corresponding to agent 1 is the topmost (solid blue) one; the curve corresponding to agent 2 is the second from the top (dashed red); and the curve corresponding to agent 3 is the lowest (dotted orange) one.

Figure 1.4 shows the optimal payments and the corresponding equilibrium payoffs as we vary the link weight G_{23} , under parameter values specified in the caption. Figure 1.4(a) depicts optimal payments to each agent as a function of G_{23} . The payment is non-monotonic in own links: once payments are nonconstant in the strength of that link, increasing G_{23} initially decreases agent 2's payment. The numerical example also illustrates a corresponding non-monotonicity in payoffs: strengthening one of an agent's links can decrease his equilibrium payoff under the optimal contract. Figure 1.4(b) depicts the equilibrium payoffs under optimal payments as a function of G_{23} . Strengthening the link between agents 2 and 3 can *decrease* the resulting payoffs for agents 1 and 2.

To give some intuition, observe that we are strengthening a link G_{23} between two agents with a common neighbor 1. When we do so, we may want to increase incentives to one of these neighbors. It turns out this agent is 3, who is now connected enough to include in the team. But the balanced neighborhood equity condition requires that as we increase τ_3 , we must keep the total equity $G_{12}\tau_2 + G_{13}\tau_3$ of 1's neighbors equal to the total equity in other neighborhoods. So to allocate more shares to agent 3 while maintaining the balance condition, the principal must either take shares away from agent 2 or give out substantially more shares to agents. The example demonstrates that the principal can prefer the former to the latter. More generally, we expect such a force to push toward non-monotonicities when we strengthen a link between two agents with one or more common neighbors (though there can be other countervailing forces).

This finding contrasts with an intuition that one might have from the network games literature, that agents are better off from becoming more central. Under fixed payments, all agents' payoffs are monotone in the network. In the present setting, however, agent 2 can benefit from weakening one of his links. There is therefore a tension between the network formation incentives of the principal and the agents. Agents may not be willing to form links that would benefit the principal or the team as a whole, even if link formation has no technological cost.

1.5 Production functions with broader complementarities

The balance condition applies beyond network games. We briefly illustrate with two standard production functions where complementarities operate at the team level rather than through bilateral network links. We show that our framework yields simple characterizations of optimal contracts in these classical settings as well and that these characterizations have implications for the amount of pay dispersion in teams.

We derive optimal contracts for Cobb-Douglas and constant elasticity of substitution production functions. The solutions describe how much incentives should be concentrated on more productive agents, and our main finding is that the optimal contract exhibits more unequal compensation when agents are more substitutable.

We work again in the simple success-or-failure environment with risk-neutral agents.¹⁵ For Cobb-Douglas production, team performance is

$$Y(\mathbf{a}) = \prod_{i=1}^n a_i^{\gamma_i},$$

where the factor shares γ_i can differ across agents. For constant elasticity of substitution (CES) production, team performance is

$$Y(\mathbf{a}) = \left(\sum_{i=1}^n \gamma_i a_i^\rho \right)^{\kappa/\rho}$$

for a non-zero ρ . The parameter κ captures returns to scale and the factor shares $\gamma_i > 0$ can again differ across agents. Finally, the elasticity of substitution between agents' efforts is $\frac{1}{1-\rho}$.

We will characterize optimal contracts when the first-order approach is valid, and can establish sufficient conditions for this:

¹⁵The assumption that all agents have the same quadratic effort cost $C_i(a_i) = a_i^2/2$ is not essential: it is straightforward to extend the subsequent analysis to heterogeneous quadratic costs by rescaling actions.

Fact 3. [Assumption 1](#) holds if $P(\cdot)$ is concave and:

- (i) For Cobb-Douglas, $\gamma_i \leq 1$ for all i and $\sum_{i=1}^n \gamma_i < 2$,
- (ii) For CES, $\rho \leq 1$ and $\kappa \in (0, 1]$.

One can directly characterize the optimal contract by applying [Theorem 1](#) (see [Zuo \[2024\]](#), which solves a closely related example in the Cobb-Douglas case). We take an alternate approach of transforming our problem to a simpler one and then applying [Theorem 1](#).

For Cobb-Douglas, consider an equivalent transformed problem in which we replace Y with $\tilde{Y} = \log(Y)$ and $P(Y)$ with $\tilde{P}(\tilde{Y}) = P(\exp(\tilde{Y}))$. The problem is now separable:

$$\tilde{Y}(\mathbf{a}) = \sum_{i=1}^n \gamma_i \log(a_i).$$

For CES, consider a transformed problem in which we replace Y with $\tilde{Y} = \frac{1}{\rho} Y^{\frac{\rho}{\kappa}}$ and $P(Y)$ with $\tilde{P}(\tilde{Y}) = P((\rho \tilde{Y})^{\frac{\kappa}{\rho}})$. Analogous to Cobb-Douglas, the problem is separable:

$$\tilde{Y}(\mathbf{a}) = \frac{1}{\rho} \sum_{i=1}^n \gamma_i a_i^{\rho}. \quad (1.6)$$

These transformations do not change the optimal contract or the corresponding equilibrium actions.¹⁶ But because the transformed problems have additively separable team performances, we can now easily calculate agents' productivities and centralities.

This lets us obtain the following characterizations of optimal contracts:

Proposition 4. *In the setting of this section, suppose [Assumption 1](#) holds.*

- (a) *For Cobb-Douglas production, the optimal payments τ_i^* are proportional to γ_i .*
- (b) *For CES production, the non-zero optimal payments¹⁷ τ_i^* are proportional to $\gamma_i^{\frac{1}{1-\rho}}$.*

¹⁶For Cobb-Douglas production and CES production with $\rho < 0$, we restrict attention to positive action profiles, the domain on which the transformations below are defined.

¹⁷If $\rho = 1$, then only agents with maximal γ_i receive positive transfers. See [Fact 4](#).

The Cobb-Douglas solution is intuitive: the principal pays more to agents with higher factor shares. Turning to CES, we see that pay dispersion under the optimal contract depends on the elasticity of substitution $\frac{1}{1-\rho}$. When this elasticity is high, so agents' efforts are highly substitutable, the optimal contract pays agents with higher factor shares much more. Conversely when the elasticity of substitution is low, so agents' efforts are highly complementary, there is little pay dispersion.

Several limiting cases of the CES result are illustrative. In the limit as $\rho \rightarrow -\infty$ with $\sum_i \gamma_i = 1$, the production function converges to the Leontief production function $Y(\mathbf{a}) = (\min_i a_i)^\kappa$. In this case, it is optimal to pay all agents equally because inducing one agent to take a higher action than others does not improve team performance. In the limit as $\rho \rightarrow 0$, again assuming $\sum_i \gamma_i = 1$, the production function approaches the Cobb-Douglas, where output is quite responsive to each agent's own effort, and contributes to transformed output in proportion to γ_i . Then compensation turns out to be linear in γ_i as in part (a) of the proposition. Finally, note that for any ρ , the relative compensation of agents does not depend on the returns to scale κ , but the total amount paid to agents when the project succeeds (and thus the absolute payments to individual agents) does depend on κ .

The Cobb-Douglas and CES cases show that the balance condition provides explicit characterizations of optimal incentives for production functions that have been difficult to analyze in the network games literature. A key point is that because we assume outcomes in our model depend on a one-dimensional team performance (via functions $P_s(Y)$ that can be quite general), we can apply monotone transformations that simplify the relevant spillovers between agents.

1.6 Concluding discussion

We have studied an incentive design problem for a team whose members contribute via non-contractable effort. We investigate how optimal contracts depend on the team's production function. Our main result is a necessary condition for contract optimality. We show that optimal contracts must satisfy a balance condition across agents receiving positive incentive pay.

In applications, our analysis offers a new perspective on compensation design that comes from focusing on spillovers. The overall theme is deriving interpretable insights into the roles of standalone capabilities and collaboration. We close with a few remarks on main features of this model and potentially important variations on them.

1.6.1 Participation constraints

The setting we have presented has no formal outside options; instead of participation constraints, it uses the constraint of limited liability to make the contracting problem nontrivial.

Nevertheless, one could take the same model of production to a setting with standard participation constraints, where some participation constraints would bind at the principal's optimum. The analogue of [Assumption 1](#) would have to be made, not for all perturbations, but in a constrained space of contract perturbations where these constraints continue to be respected (as in [Section A.2.2](#)). Moreover, the balance condition would need to be augmented with agent-specific Lagrange multipliers on the binding constraints. If one agent works harder, this relaxes or tightens the constraints for other agents, and the balance condition would involve the multipliers on the constraints to account for these spillovers. We leave a full treatment of this case to future work.

1.6.2 Robustness to measurement error

Our analysis has assumed that the principal knows the network of incentive spillovers. In practice this network would be estimated with error. Suppose a principal takes measured spillover parameters at face value, checks the balance condition at a given contract, and perturbs incentive pay when it fails. [Section A.3](#) takes up the question of when this strategy can work.

It turns out that certain spectral statistics of the network play a subtle role in the answer. When spillovers are moderate—the spectral norm of the effective spillover matrix is bounded away from its critical value—the balance condition is robust to measurement error as long as this error is not too large. By contrast, when spillovers are strong and the team is nearly partitioned into weakly connected subteams, even arbitrarily small measurement error can mislead the principal about which subteam is more responsive to incentives. We can rule out

this issue when the network’s spectral gap, a standard measure of segregation [von Luxburg, 2007, Golub and Jackson, 2012], is bounded away from zero. This extension connects our framework to the econometrics of networks and to recent work on interventions with noisy spillover estimates [Galeotti et al., 2025].

1.6.3 Other objectives

While we have focused on a standard principal concerned with maximizing profits, the analysis of how contract perturbations affect team performance is equally relevant for understanding other natural principal objectives. For example, the objective could be maximizing a (weighted) sum of workers’ utilities. The key insight is that, under suitable conditions, the first-order conditions of such an optimization problems involve only the local comparative statics of the equilibrium, and the incentive spillover forces we discuss would be equally relevant.

1.6.4 The role of a one-dimensional outcome

An important simplifying assumption throughout the analysis is that actions influence outcomes only via a one-dimensional team performance $Y(\mathbf{a})$. Moving beyond this assumption to settings where probabilities of outcomes depend in an arbitrary way on the full action profile is a natural direction. Though a first-order approach may continue to be fruitful, establishing conditions for its validity is likely challenging. Subject to that, we expect that generalizations of our balance conditions would hold. However, the outcome distribution may now provide more fine-grained information about an agent’s effort, and the principal will use this information to design optimal incentives (as in, e.g., Holmström [1982] and Legros and Matthews [1993]). The generalized balance conditions must incorporate this along with agents’ centralities and productivities. So the balance conditions would be suitably adjusted, and rankings such as those in Section 1.3.3 would also be affected by these informational considerations.

Chapter 2

Equity Pay In Networked Teams

2.1 Introduction

In this chapter, we further study properties of the optimal contract in the linear-quadratic setting of [Section 1.4.2](#). The analysis in that section focuses on the intensive margin by characterizing equity payments among those who receive a positive payment. To characterize optimal contracts fully, however, we must also optimally choose the active set: the set of agents that get positive equity. It turns out that it need not be optimal to give all agents positive pay and the optimal team induced to work may be a subset of the potential contributors. We study the problem of choosing this optimal team. Then, armed with a characterization of optimal equity allocations on both the intensive and extensive margins, the third set of results focuses on some implications of applied interest, examining how equity shares and outcomes vary with the network and the strength of complementarities. We briefly describe our contributions below.

Extensive margin.

We approach the problem by first reducing it to a quadratic program. This allows us to deduce two results implying considerable structure on the active set. Substantively, we find that highly connected subnetworks are optimal for the principal. Our first result on this

is that the active set under any optimal allocation always has diameter at most two in the complementarity network, meaning that any two active agents either have complementarities with each other or both have complementarities with some shared active neighbor. Our interpretation of this result is that an optimal active set should be sufficiently “tight-knit.” We give a rough intuition for this result. When equity is given to members of a tight-knit group, the incentives given to one member also motivate effort by the others due to spillovers. On the other hand, if a given amount of equity is given to two subsets of agents that are not tightly linked in the complementarity network, then the equity given to one subset dilutes the incentives of the other without a strong counteracting beneficial effect of spillovers. Thus, a principal prefers to “concentrate” incentives and focus them on a single highly complementary group. This force is seen even more sharply in our second main result in this section, when we restrict attention to the standard benchmark of unweighted networks—in which all non-zero complementarities have the same strength. In this case, any maximum clique (a subnetwork with complementarities between all pairs of agents) is an optimal active set.

Our overall interpretation of these results is that when a firm relies on a single joint outcome to provide incentives, teams with dense or tightly-knit complementarities outperform more dispersed teams. A further implication is that the principal often prefers to make a small team exert large efforts in order to make the most use of complementarities, rather than eliciting less effort from a larger group with more diffuse complementarities. Under convex effort costs, this can entail a considerable loss in agent welfare compared with other policies yielding somewhat less output.

Implications.

Our final set of results explores some implications of our theoretical characterizations motivated by applied questions about the structure of teams and their compensation.

An important set of questions concerns how optimal equity pay and team performance depend on the network and the strength of complementarities. Our characterization lets us calculate explicitly how the optimal shares vary as the network changes. These comparative statics show that payments can behave quite differently from measures such as Bonacich

centrality, relevant in network games with exogenous incentives. In particular, monotonicity properties, whereby strengthening one's network links necessarily increases centrality, do *not* hold for payments. This generalizes the observation made for the network on three agents in [Section 1.4.2](#) (see [Figure 1.3](#)).

Questions of network design are interesting more generally. Which links are most valuable to the principal? To make some progress toward understanding this issue, we ask how strengthening links between agents affects the team's probability of a successful outcome. The value to the team of strengthening a link takes a surprisingly simple form: it is proportional to the product of the equity shares allocated to the two agents. So if the principal can strengthen some complementarities, it is most valuable to focus on connections between agents who are already (equity) central.

A final, practically important, question is how much equity a firm should devote to compensation. The trade-off is that allocating more equity to the team elicits more effort from them, but the principal gets a smaller share of the resulting pie. How a principal manages this trade-off depends on the environment, including the network and the strength of complementarities. Our last result gives conditions under which a profit-maximizing firm wants to distribute more shares to agents when complementarities are stronger. Intuitively, greater complementarities in production increase the returns to allocating shares to agents, since each share now drives more additional effort through spillovers. Establishing this for general complementarity networks turns out to be subtle.

2.2 Objectives

In addition to the residual profits objective considered in [Chapter 1](#), we also analyze optimal contracts for the *success probability* objective. Under this, the principal maximizes the probability of success $P(Y(\mathbf{a}^*))$ subject to the constraint that the total transfers are no larger than the output from the project. This constraint rules out positive transfers when the project fails. Thus, the success probability maximization problem is

$$\text{choose } \boldsymbol{\tau} \text{ to maximize } P(Y(\mathbf{a}^*)) \text{ subject to } \sum_i \sigma_i \leq 1. \quad (\text{SP})$$

An alternative model, which turns out to be very similar, is that the project yields a deterministic monetary output $P(Y)$, where $P : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a strictly increasing, concave, and twice differentiable function. Our analysis applies essentially unchanged to the study of this production function with either objective, provided that the principal uses *linear* contracts, i.e., giving each agent a transfer equal to a fixed equity share σ_i of output.¹ In the rest of the paper, we will work with the binary-outcome model.

2.3 The extensive margin: Active and inactive agents

We ask which agents are active and which agents are inactive under optimal contracts for a given complementarity network. Recall an agent is defined to be active under a given allocation if he receives positive equity.

The main results in this section show that the active sets under optimal allocations are highly connected subnetworks. We first show that any optimal active set has diameter at most two in the complementarity network $\mathbf{G}$. We then show that in the special case of unweighted networks, there is an optimal allocation with a clique as the active set. We interpret these results as saying that, when incentives to exert effort are based only on global outcomes (and not local measures of performance), smaller and more highly connected teams outperform larger and more dispersed teams.

It is not immediate from [Proposition 2](#) which agents are active; indeed, there can be several candidate active sets compatible with the condition of the theorem. ([Proposition 2](#) implies that the candidate active sets are the subnetworks $\tilde{\mathbf{G}} \subseteq \mathbf{G}$ such that the row sums of $\tilde{\mathbf{G}}^{-1}$ are positive.)

The key to our analysis is the fact (formalized as [Lemma 14](#) in [Section B.1](#)) that we can find the active sets under optimal allocations by solving the following optimization problem²

¹If P is not bounded, this deterministic model requires an additional assumption that β is small enough so that the principal's feasible payoffs are bounded.

²This is a quadratically constrained optimization of a linear objective, since the constraint can be rewritten as $\sigma_i(\mathbf{G}\boldsymbol{\sigma} - \mathbf{c})_i = 0$.

among non-negative equity shares summing to a fixed value:

$$\begin{aligned} & \max_{\tau} \quad c \\ & \text{subject to} \quad (\mathbf{G}\boldsymbol{\sigma})_i = c \text{ whenever } \sigma_i > 0. \end{aligned} \tag{2.1}$$

The reason for this reduction is that, when the balanced equity condition holds, we can rewrite output as the following function of the total shares allocated to agents and c , the total weighted equity in each active agent's neighborhood:

$$Y(\mathbf{a}^*) = \left(\sum_{i \in N} \tau_i \right) \left(\frac{P'(Y^*)}{1 - \beta P'(Y^*)c} + \frac{\beta P'(Y^*)^2 c}{2(1 - \beta P'(Y^*)c)^2} \right). \tag{2.2}$$

The right-hand side is an increasing function of c . So the principal can choose an optimal active set by maximizing the constant c in the balanced equity condition.

One important implication of this is the following invariance of the principal's optimization to the complementarity parameter:

Proposition 5. *If $\boldsymbol{\sigma}_0^*$ is a solution to the principal's problem under $\beta_0 > 0$ and β_1 is another complementarity parameter, then there is a constant $k > 0$ so that $k\boldsymbol{\sigma}_0^*$ solves the principal's problem under β_1 .*

This says essentially that the active set and the ratios in which equity is optimally allocated are both independent of β . This follows from observing that the optimization problem to solve for the active set does not depend on β . For the success probability objective, the constant k is equal to 1. Under the residual profit objective, the principal may adjust the total share of output distributed to agents as β changes (see [Section 2.4.2](#)).

Before turning to other general implications of the lemma, we study active set considerations in the example of a three-agent network. This example, which is the smallest interesting case of our model, shows that the active set can depend in non-trivial ways on network structure. We describe the optimal allocation here and provide details in [Section B.2](#).

Example 1. Consider a weighted network with three agents without self-links (see [Fig-](#)

ure 1.3). Since it is optimal to have both agents active in networks with two agents, three-agent networks are the smallest non-trivial example of our model.

Without loss of generality, we can assume $G_{12} \geq G_{13} \geq G_{23}$ and choose the normalization $G_{12} = 1$, so that the adjacency matrix is

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & G_{13} \\ 1 & 0 & G_{23} \\ G_{13} & G_{23} & 0 \end{bmatrix}.$$

The optimal active set consists of either agents 1 and 2 or all three agents. If all agents are active, then Proposition 2 implies that the optimal shares must solve

$$\tau_1^* = \frac{1 + G_{13} - G_{23}}{2G_{13}}c, \quad \tau_2^* = \frac{1 + G_{23} - G_{13}}{2G_{23}}c, \quad \tau_3^* = \frac{G_{23} + G_{13} - 1}{2G_{23}G_{13}}c,$$

for some constant c . Since we must have $\sigma_3^* > 0$ for all agents to be active, a necessary condition for $\{1, 2, 3\}$ to be an optimal active set is $G_{13} + G_{23} > 1$.

This necessary condition also turns out to be sufficient; we now sketch the argument for this. A calculation shows that distributing a total amount s of equity consistent with these ratios gives a value of

$$c = \frac{2G_{13}G_{23}s}{2(G_{23} + G_{13}) - 1 - (G_{13} - G_{23})^2}.$$

On the other hand, if a total amount s of equity is distributed among two active agents, then $c = s/2$. A bit more algebra confirms that

$$\frac{2G_{13}G_{23}s}{2(G_{23} + G_{13}) - 1 - (G_{13} - G_{23})^2} > \frac{s}{2}$$

whenever $G_{13} + G_{23} > 1$.

So the active set includes all three agents if and only if $G_{13} + G_{23} > 1$. In this example, the principal maximizes $(\mathbf{G}\boldsymbol{\tau})_i$ subject to the balanced equity condition by choosing an active set that maximizes the minimum weighted degree of an agent in the induced subnetwork. If the least connected agent's complementarities are too weak, the principal prefers to exclude

that agent from the team and concentrate on incentivizing the two agents with stronger complementarities.

Moving back to general networks, we now state two results showing that the principal prefers a highly connected active set. The first result on this holds for any network satisfying our maintained assumptions. It states that the network distance between any pair of agents in the active set is small. Recall that the diameter of a network is the longest distance³ between any two agents in the network.

Proposition 6. *The diameter of the active set under any optimal allocation is at most 2.*

The idea is that if two agents i and j are at distance larger than two from each other, their neighborhoods are disjoint. An optimal allocation must divide shares between the disjoint sets $\{i\}$, $\{j\}$, and $N(i)$, and $N(j)$ (as well as any other potential agents in the active set). It must also satisfy the balanced neighborhood equity condition, which implies that the shares allocated $N(i)$ and $N(j)$ cannot be very large. Indeed, the proof shows that any such allocation is dominated by allocating shares to only two agents: splitting shares evenly between two agents connected by a link with the largest weight in the network gives a higher value of the constant c .

In the special case of unweighted networks, the message that highly connected active sets are optimal can be sharpened: It is an optimal solution for the principal to choose any clique⁴ of maximum size and divide shares equally among the agents in this clique.

Theorem 3. *If G is an unweighted network, then any maximum clique is the active set at an optimal allocation.*

When connections are unweighted, choosing a subset that is as densely connected as possible leads to at least as high a payoff as choosing a larger but more sparsely connected subset, even if all agents in the larger subset have higher degree.

The proof applies [Lemma 14](#): given an optimal allocation with an arbitrary active set, we find a clique within that active set for which the constant c is as large as possible. We

³Shortest path consisting of links with positive weights.

⁴A subnetwork with links between all pairs of agents.

produce such a clique by sequentially constructing a series of agents who are all connected to each other. To do so, the balanced equity condition for the optimal allocation has to be used carefully at each step of the construction.

In general, the active set need not be unique in unweighted networks, so there can be other optimal allocations giving the same payoff to the principal as a clique of maximum size. For example, in a star network, any set including the central node and at least one peripheral node is the active set at some optimal allocation.⁵ The next example shows that there can also be optimal allocations that differ more substantially from maximum cliques.

Example 2. Consider an even number of agents n arranged in a circle. Let $\mathbf{G}$ be the undirected network in which each agent is connected to all other agents except the diametrically opposite agent in the circle: for all distinct i and j we let $G_{ij} = 1$ if $|i - j| \neq n/2$ and $G_{ij} = 0$ if $|i - j| = n/2$. The network structure is shown in [Figure 2.1](#).

Suppose that we want to allocate s shares to agents optimally. The maximum cliques have size $n/2$, and dividing the s shares evenly within any maximum clique gives $(\tilde{\mathbf{G}}\boldsymbol{\tau})_i = \frac{n-2}{n} \cdot s$. All agents in the full network have degree $n - 2$, so dividing s shares evenly among all agents also gives $(\tilde{\mathbf{G}}\boldsymbol{\tau})_i = \frac{n-2}{n} \cdot s$. It follows that the set of all agents, as well as all maximum cliques, are possible active sets, depending on which optimal allocation is chosen.

This example shows that the principal can be indifferent between very different active sets—a point which has implications for the welfare of agents, as we discuss further in [Section 2.5.2](#).

2.4 Implications

In this section, we explore some implications of our analysis that illuminate how the optimal contract depends on the environment. We focus on the effects of changes in the network $\mathbf{G}$ and the parameter β describing the strength of complementarities. [Section 2.4.1](#) examines how the equity allocation and the resulting team performance depend on the network of complementarities. The results provide some insights about which networks might be preferred by the principal and by agents. [Section 2.4.2](#) then asks how the share of equity retained

⁵Among allocations giving s shares to agents, a given allocation is optimal if and only if it gives $s/2$ shares to the central agent and $s/2$ shares to the peripheral agents.

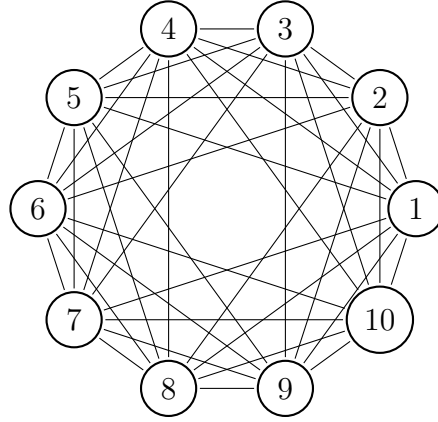


Figure 2.1: Ten agent unweighted graph with each agent connected to all other agents except the diametrically opposite one.

by the principal (in the residual profit maximization problem) depends on the strength of complementarities.

2.4.1 Varying the network

Our first result describes how optimal allocations vary as the network changes. We write $\frac{\partial}{\partial G_{jk}}$ for the derivative in the weight $G_{jk} = G_{kj}$ of the link between j and k . Recall that given an allocation, we write $\tilde{\mathbf{G}}$ for the adjacency matrix restricted to active agents.

Proposition 7. *Suppose that under $\mathbf{G}$ there is a unique optimal equity allocation⁶ $\boldsymbol{\tau}^*$, with agents i , j , and k all active. The derivative of agent i 's optimal share as we vary the weight of the link between j and k is*

$$\frac{\partial \tau_i^*}{\partial G_{jk}} = -(\tilde{\mathbf{G}}^{-1})_{ik} \sigma_j^* - (\tilde{\mathbf{G}}^{-1})_{ij} \sigma_k^* + \frac{\partial c}{\partial G_{jk}} \frac{\sigma_i^*}{c}.$$

The value c is the balanced equity in each neighborhood from [Proposition 2\(a\)](#). The proof is based on the matrix calculus expression

$$\frac{\partial \mathbf{G}(t)^{-1}}{\partial t} = -\mathbf{G}(t)^{-1} \frac{\partial \mathbf{G}(t)}{\partial t} \mathbf{G}(t)^{-1} \quad (2.3)$$

⁶We expect this hypothesis to be satisfied for generic networks.

for the derivative of the inverse of a matrix.

The result provides a fairly explicit expression for the impact of changing a link on equity allocations. However, calculating the change $\frac{\partial c}{\partial G_{jk}}$ in c may be difficult under the residual profits objective, where the total amount of equity allocated can change as we strengthen a link. We can be more explicit under the success probability objective, since the total amount of equity allocated sums to 1. In that case, we have an explicit expression

$$c = \frac{1}{\mathbf{1}^T \tilde{\mathbf{G}}^{-1} \mathbf{1}}$$

for the total equity in each neighborhood. Differentiating this expression gives a version of [Proposition 7](#) without an unknown value c .

Under either objective, the ratio $\sigma_i^*/\sigma_{i'}^*$ between two agents' shares is independent of c , so (2.3) lets us calculate the change in this ratio (under either objective) as the link between j and k is strengthened.

The matrix inverse $\mathbf{G}^{-1}$ need not vary monotonically as $\mathbf{G}$ changes. This implies that equity need not satisfy monotonicity properties that hold for standard centrality measures such as Bonacich centrality.

We next look at how team performance under an optimal allocation varies as the network changes. Recall that Y^* denotes the equilibrium team performance under an optimal allocation. Then $\frac{\partial Y^*}{\partial G_{ij}}$ is the change in this team performance as the weight on the link between agent i and j increases.

Proposition 8. *Suppose $\boldsymbol{\tau}^*$ is an optimal allocation. Then the change in equilibrium team performance as G_{ij} varies can be expressed as*

$$\frac{\partial Y^*}{\partial G_{ij}} = \sigma_i^* \sigma_j^* h,$$

where h does not depend on the identities of i or j .

The proposition says that the increase in output from strengthening a link is precisely pro-

portional to the product of the equity shares given to the two agents connected by that link. The proof gives an explicit formula for the quantity h , which depends on the model parameters and the allocation.

The proposition has implications for a designer who can make small changes in the network of complementarities. If the principal could marginally strengthen some links, she would want to focus on links between pairs of agents with high equity centralities. This is consistent with the intuition from [Section 2.3](#) that highly connected groups of agents are especially productive under equity compensation.

2.4.2 Varying complementarities

We now turn to how outcomes change as the complementarity parameter β increases. Recall from [Proposition 5](#) that ratios of optimal shares do not depend on the value of β . But under the residual profits objective, we can ask how the total fraction of shares allocated to agents depends on β .

We study the comparative static in the special case when $P(\cdot)$ is linear in the range of feasible team performances. We assume for simplicity that the optimal allocation is unique, but could easily relax this assumption. The principal faces a trade-off between keeping a larger share of the profits and using a larger share to encourage workers to exert more effort. The following result states that when complementarities in production are larger, it is optimal to keep a smaller share of a larger pie.

Proposition 9. *Suppose that $P(Y) = \alpha Y$ on an interval $[0, \bar{Y}]$ containing the equilibrium team performance under any feasible allocation and that there is a unique optimal allocation τ^* . Under the residual profits objective, the sum of agents' equity shares under the optimal allocation is increasing in β , i.e.,*

$$\frac{\partial (\sum_{i \in N} \tau_i^*)}{\partial \beta} > 0.$$

The basic idea behind the proof is that the benefits to retaining more of the firm are linear in the output while the benefits to allocating more shares to workers are convex, and become

steeper as complementarities increase.

If $P(Y)$ is strictly concave, there is a trade-off between the concavity of $P(Y)$ and the convexity of $Y(\mathbf{a})$. Depending on which effect is stronger, the fraction of shares allocated to agents may increase or decrease as complementarities grow stronger.

2.5 Discussion

2.5.1 Tightly-knit teams

Our extensive margin results can be summarized as saying that the principal prefers to concentrate equity in teams whose members have strong mutual complementarities. In terms of interpretation, this need not mean that the agents involved work closely together or are nearby in an organizational sense—just that their efforts are highly complementary in producing the output.

The details of the extensive margin results depend on the specific structure of our model, but we believe the economic intuition underlying these results has broader implications. When the principal motivates an agent by giving him a larger share of the equity in a single common output, it dilutes the equity of the others. Strong complementarities among those getting equity shares countervail this dilution, and this is what makes tight-knit teams valuable to the principal. As we remarked in our discussion of the model's assumptions, in reality a principal may have signals of effort richer than we have studied—for example, outcomes reflecting contributions of specific organizational units. It may be interesting to study how a principal would optimally use multiple signals of this sort to provide incentives to a networked team.

2.5.2 A tension between the principal's interests and agent welfare

[Example 2](#) shows that the principal can be indifferent between active sets, and associated allocations of equity, that have very different welfare implications for workers. In that example, since the cost of effort is convex, agents are better off (on average) if the same performance is achieved by a larger team. But the principal is indifferent between two different team sizes. If the complementarities are perturbed slightly to strengthen those

in some maximum clique, then the principal’s indifference is broken and she has a strict preference for the compensation scheme that motivates a smaller team—and which happens to leave workers substantially worse off.

This is a consequence of the fact that, subject to paying out a certain total share in equity compensation, the principal is maximizing the probability of a project’s success rather than utilitarian welfare. The mechanism of equity pay can do a very poor job of transmitting workers’ interest in a more equal distribution of effort. This highlights an interesting tension between welfare and the principal’s preferred mode of incentive-provision, and the binary-outcome model we work with brings it out particularly sharply.

2.5.3 Connection to a spectral radius maximization problem

The optimal allocation turns out to have a simple description in terms of a problem of maximizing a spectral radius: if σ solves the success probability optimization problem, then $\Sigma = \text{diag}(\sigma)$ also maximizes the spectral radius $\rho(\mathbf{T}\mathbf{G})$ among nonnegative vectors σ summing to 1.⁷ To show this, we show that when β is large enough (so that very large spillovers are possible), the principal wants to choose shares τ inducing a large spectral radius to capture these spillovers. By [Proposition 5](#), the optimal allocations do not depend on β , so in fact such a σ is optimal for any β . We formally state and prove the connection in [Section B.3](#).

An applied mathematics literature discusses spectral radius maximization problems of this form (e.g., [Elsner and Haderer \[2015\]](#), [Nesterov and Protasov \[2013\]](#), and [Axtell, Han, Hershkowitz, Neumann, and Sze \[2009\]](#)). Most closely related, [\[Elsner and Haderer, 2015\]](#) consider the same spectral radius maximization problem and discuss algorithms for efficiently computing the optimal diagonal matrix $\mathbf{T}$. Our analysis turns out to provide several insights into this problem. For instance, [Theorem 3](#) implies a characterization of the highest achievable spectral radius when $\mathbf{G}$ is the adjacency matrix of an unweighted network, showing that it is achieved by dividing shares σ_i equally among the members of a maximum clique.

⁷The spectral radius of a matrix, which we denote by $\rho(\cdot)$, is the largest magnitude of an eigenvalue of the matrix.

Chapter 3

Network Interventions: Targeting Agents or Targeting Links?

3.1 Introduction

Many economic settings feature peer effects between connected agents. As one example, consider firms deciding how much to invest in research and development in the presence of a network of R&D collaborations.¹ Another leading example is students deciding how much to study given their social networks.² Following a large literature, we model such settings as network games with complementarities: each agent chooses a level of effort, and an agent's effort is more productive when neighbors exert more effort [Ballester, Calvó Armengol, and Zenou, 2006].

When network spillovers are important, a natural question is how to target interventions given these spillovers (Galeotti, Golub, and Goyal [2020]). Policymakers often implement

¹Research and development collaborations are an important driver of innovation (see, for example, Belderbos, Carree, and Lokshin, 2004). Motivated by this, a theoretical literature models firms' choices about R&D investment as a network game (see, for example, Goyal and Joshi, 2003, Goyal and Moraga-González, 2001, and König, Liu, and Zenou [2019]).

²A large literature analyzes peer effects from friends in a social network (e.g., Bramoullé, Djebbari, and Fortin [2009] and Calvó-Armengol et al. [2009]). Much of this work treats links as exogenous, but links may be endogenous in practice [Goldsmith-Pinkham and Imbens, 2013].

interventions incentivizing individual agents. But other types of interventions target links between agents. Consider, for instance, a government trying to subsidize R&D. One approach is to target individual effort with subsidies or related policies, e.g., subsidizing R&D at particular firms. Another is to create or subsidize links between agents, e.g., subsidizing collaborative efforts between firms or R&D clusters. We examine how these two intervention strategies interact and identify conditions under which one is more effective than the other.

Our model is a network game of strategic complementarities with endogenous link formation. Agents simultaneously choose a costly action and choose, for each other agent, a costly bilateral link intensity, corresponding to investing in connecting with an agent. These link intensities determine a network of spillovers, where the symmetric strength of a link between a pair of agents is the total of link intensities. An important parameter, which we call the *baseline link incentive* (denoted by s_{ij}), describes whether a link has intrinsic value. To provide an interpretation of this quantity, if the incentive is non-negative ($s_{ij} \geq 0$), agent i forms a link even when agent j takes no action. If it is negative ($s_{ij} < 0$), a link is formed only when agent j takes a sufficiently large action.

A planner intervenes on this environment by providing monetary subsidies to agents for exerting effort and forming links. The common monetary units lets us compare link subsidies to action subsidies. The planner seeks to maximize a welfare objective when agents play an equilibrium in the resulting game, subject to a budget constraint on the subsidy payments. The objective can be any arbitrary increasing function of agents' actions. An example is the sum of all agents' actions — which could correspond to a policymaker targeting aggregate investment in R&D or the total time students devote to studying. What will the planner focus on: action subsidies or link subsidies?

These two instruments affect quantities relevant to the planner — welfare and planner payments — through different spillover channels. To illustrate this, suppose the planner increases the action subsidy of a given individual, which increases his action. Because actions are strategic complements, his neighbors increase their actions. Under endogenous link formation, the individual forms stronger links with his neighbors as a result of this increase in actions. Then, the neighbors of these neighbors increase their actions and form stronger links and so on and so forth. By contrast, suppose the planner increases the link subsidy for

a given pair of individuals, which strengthens their link. A stronger link increases their effort incentives, thereby increasing their individual actions. Their own neighbors, neighbors of these neighbors and so on, then increase actions and form stronger links. Any optimal intervention must compare the welfare and payment effects of these two different propagation channels.

Our main result describes qualitative properties of an optimal intervention (subject to regularity conditions). It turns out the key determinant of these properties is the baseline incentive to form links. If a pair of agents has a non-negative baseline incentive to form links ($s_{ij} \geq 0$), the optimal policy does not subsidize their link. By contrast, if two connected agents have a negative baseline incentive ($s_{ij} < 0$), then the optimal policy will not subsidize both of their actions without subsidizing their link. Notably, these qualitative properties of the optimal intervention are not sensitive to how the costs to agents of exerting effort and forming links compare.

The basic insight is that the planner should not subsidize connections that have inherent value to the agents, but may want to subsidize connections between welfare-relevant agents who are not inclined to connect. In the education setting, for example, the result cautions against policies encouraging connections between socially compatible students. Connecting students who would not interact for non-academic reasons, however, can be optimal. This point might be straightforward in a model with binary link choices, but is less obvious when agents choose the intensity of connections.

To give some intuition for the result, if the network is sufficiently connected, then (as we describe below) the benefits to strengthening links can be very large. Moreover, the link formation protocol creates free-riding incentives between pairs of agents. But it is nevertheless cheapest for the planner to reward high actions and let agents form stronger links in response. When baseline incentives are negative, however, agents' underinvestment in link formation can become more severe, so the planner can optimally target this margin by directly subsidizing links.

We compare the implications of the endogenous link formation model to those of a benchmark model with exogenous links and find the basic structure of the optimal intervention reverses. In this benchmark, agents first make link-formation decisions without considering

the subsequent game (which is the relevant object for the planner’s objective) and then play this game. This is equivalent to a setting where links are exogenous, and we choose this formulation so that the planner’s interventions to strengthen links can be implemented via subsidies. We begin by finding that when spillovers are large, the optimal intervention does not subsidize actions for a pair of connected agents without subsidizing their link. In the large spillover regime, subsidizing links can have unboundedly higher benefits than subsidizing actions. Intuitively, individual subsidies have a linear effect on actions while the effects of link subsidies can scale much faster than linearly. So the returns to investing in highly connected networks can be very high. We next find that the optimal intervention focuses more on subsidizing actions when agents’ actions are low, as occurs when spillovers are weak. Both of these conclusions stand in contrast to the endogenous case. In that setting, the planner does not subsidize links when agents’ actions are high — corresponding to non-negative s_{ij} — but can optimally subsidize links when agents’ actions are low, as induced by negative s_{ij} .

Finally, we ask to what extent our main result generalizes beyond quadratic settings. We identify conditions on agents’ utilities driving each part of our result. When baseline incentives to form links are positive, sufficiently convex link formation costs and sufficiently concave action complementarities imply that the optimal policy does not subsidize links. By contrast, when baseline incentives to form links are negative, if link formation costs are not too convex and action complementarities are sufficiently convex, the planner will not subsidize two connected agents’ actions without subsidizing their link.

3.1.1 Related Literature

There are large literatures on network games in economics, computer science, and related fields (see [Menache and Ozdaglar \[2011\]](#) and [Jackson and Zenou \[2015\]](#) for overviews). We begin with a framework with linear best responses, which is widely used for its tractability (e.g., [Ballester et al. \[2006\]](#) and [Bramoullé, Kranton, and D’amours \[2014\]](#)), and focus on the case when actions are complementary. This model has been applied to a wide range of empirical settings, including our main examples above of education ([Calvó-Armengol, Patacchini, and Zenou \[2009\]](#)) and research and development ([König, Liu, and Zenou \[2019\]](#)). Our endogenous network model is closest to parts of this literature incorporating network

formation (e.g., [Sadler and Golub \[2021\]](#)).

More recent work has explored optimal interventions in network games. Most of this literature focuses on targeting individual behavior via subsidies for effort, prices, or related instruments (e.g., [Candogan, Bimpikis, and Ozdaglar \[2012\]](#), [Bloch \[2016\]](#), [Galeotti, Golub, and Goyal \[2020\]](#), [Parise and Ozdaglar \[2023\]](#)). A smaller group of papers focuses on interventions targeting links ([Belhaj, Bervoets, and Deroïan \[2016\]](#) and [Li \[2023\]](#)) and gives conditions when the optimal networks belong to a particular class: nested split graphs. We allow both types of interventions and focus on the tradeoffs between the two.

Most closely related, [Sun et al. \[2023\]](#) and [Kor et al. \[2025\]](#) also consider interventions when the planner can target individuals and links. [Kor et al. \[2025\]](#) extend the tools from [Galeotti et al. \[2020\]](#) to settings where the planner can apply their budget to subsidizing actions or changing links. This gives a characterization of the optimal intervention in terms of a spectral decomposition of the adjacency matrix of the original network.³ The main result in [Sun et al. \[2023\]](#) is an equivalence between the two types of interventions: any intervention targeting only links induces the same outcomes as some intervention targeting only individuals. In these papers, the planner has separate budgets for subsidizing actions and changing links and the two types of interventions are not directly comparable. We instead allow endogenous network formation, which lets us compare monetary subsidies, and find optimal interventions can look quite different.

A bit further afield, there is a long line of work on interventions in network settings other than network games. Important applications include diffusion of information in networks (see [Kempe et al. \[2003\]](#) and many subsequent papers) and polarization of opinions in networks (e.g., [Gaitonde et al. \[2020b\]](#) and [Musco et al. \[2018\]](#)). We focus on a network games framework meant to capture decisions in settings with peer effects. This framework turns out to avoid some of the combinatorial challenges associated with determining optimal interventions in other network settings.

³Another difference from the present work is a quadratic cost of providing subsidies. We consider a more traditional budget constraint and find this leads to starker structure for the optimal intervention.

3.2 Model

We analyze a network game with endogenous link formation. Consider a set of n agents, who we denote by $\mathcal{N} = \{1, 2, \dots, n\}$, and a social planner. (We will use the pronouns “he” to refer to an agent and “she” to refer to a planner.)

Each agent i chooses a real-valued action $a_i \geq 0$ and a real-valued link intensity $g_{ij} \geq 0$ with each other agent $j \in \mathcal{N} \setminus \{i\}$. The link intensities between i and j determine the strength $G_{ij} = G_{ji} = g_{ij} + g_{ji}$ of a weighted, undirected link between i and j . The collection of links can be represented as a non-negative matrix $\mathbf{G} \in \mathbb{R}_{\geq 0}^{n \times n}$, with zero entries $G_{ii} = 0$ on the diagonal.

The utility to agent i from action profile $\mathbf{a}$ and link intensities $\mathbf{g}$ is

$$U_i(\mathbf{a}, \mathbf{g}) = b_i a_i + \rho \sum_{j \in \mathcal{N} \setminus \{i\}} G_{ij} a_i a_j - \frac{a_i^2}{2} + \sum_{j \in \mathcal{N} \setminus \{i\}} s_{ij} G_{ij} - \sum_{j \in \mathcal{N} \setminus \{i\}} \frac{f_{ij} g_{ij}^2}{2}.$$

We describe each term in the utility function:

- The first term is a linear individual payoff $b_i \in \mathbb{R}$ to actions which we call the standalone baseline incentive for the action. Here, the incentive b_i lies in $[\underline{b}, \bar{b}]$ where $\bar{b} > \underline{b} > 0$.
- The second term captures spillovers between agents’ actions. The parameter $\rho \in \mathbb{R}_{>0}$ describes the global strength of complementarities. Each summand $G_{ij} a_i a_j$ implies that there are complementarities between agents i and j , and these complementarities are stronger when the link G_{ij} is larger.
- The third term is a quadratic cost $C_i(a_i) = a_i^2/2$ of actions.⁴
- The fourth term is an intrinsic payoff $s_{ij} \in \mathbb{R}$ to links, which we call the standalone baseline incentive for a link. Here, the incentive s_{ij} lies in $[-\bar{s}, \bar{s}]$ where $\bar{s} > 0$.
- The final term is a quadratic cost $F_{ij}(g_{ij}) = f_{ij} g_{ij}^2/2$ to link intensities, where the parameter $f_{ij} \in \mathbb{R}_{>0}$ describes its curvature.

⁴It is not essential that the cost of actions take the particular quadratic form $C_i(a_i) = a_i^2/2$. All of the results hold when $C_i(a_i) = c_i a_i^2/2$ for any curvature $c_i > 0$.

We assume that the intrinsic payoff from links s_{ij} and the curvature of link intensity costs f_{ij} are symmetric across pairs, that is, $s_{ij} = s_{ji}$ and $f_{ij} = f_{ji}$. This symmetry arises when these primitives depend only on pairwise characteristics, such as similarity between agents or the distance separating them. The model allows the various standalone incentives and costs to be heterogeneous across agents and across links.

The planner maximizes an objective function $\mathcal{W}(\mathbf{a})$. We assume this objective is a function of agents' actions and that it is strictly increasing and differentiable in each of its arguments. She can intervene through monetary subsidies rewarding agents' actions and links. Specifically, agent i receives a subsidy $\beta_i \geq 0$ for exerting effort. Furthermore, agent i receives a subsidy $\sigma_{ij} \geq 0$ for forming a link G_{ij} with agent j . For any realized profile $(\mathbf{a}, \mathbf{g})$, the total payment made by the planner is

$$\mathcal{P} = \sum_{i \in \mathcal{N}} \beta_i a_i + \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} \sigma_{ij} G_{ij}.$$

The planner has a budget $\mathcal{B} > 0$, which can be allocated between action and link subsidies. If the planner chooses subsidies $(\boldsymbol{\beta}, \boldsymbol{\sigma})$, then agent i 's utility is

$$\mathcal{U}_i = U_i(\mathbf{a}, \mathbf{g}) + \beta_i a_i + \sum_{j \in \mathcal{N} \setminus \{i\}} \sigma_{ij} G_{ij}.$$

If the planner offers an action subsidy of β_i per unit of effort, then she will pay $\beta_i a_i$ to agent i for taking an action a_i . Similarly if she offers a subsidy of σ_{ij} for link G_{ij} , then she will pay $\sigma_{ij} G_{ij}$ to agent i for a link strength G_{ij} .

Once the planner commits to an intervention $(\boldsymbol{\beta}, \boldsymbol{\sigma})$, agents respond by playing a pure strategy Nash equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$. If no equilibrium exists under a particular subsidy scheme, we set the welfare $\mathcal{W}$ under that subsidy scheme to $-\infty$. We analyze the planner's problem under a given equilibrium selection $(\mathbf{a}^*(\boldsymbol{\beta}, \boldsymbol{\sigma}), \mathbf{g}^*(\boldsymbol{\beta}, \boldsymbol{\sigma}))$. Her problem is therefore to

maximize the objective $\mathcal{W}(\cdot)$ under this selection, subject to a budget constraint:

$$\begin{aligned} & \sup_{(\beta, \sigma)} \mathcal{W}(\mathbf{a}^*), \\ \text{subject to } & (\mathbf{a}^*, \mathbf{g}^*) = (\mathbf{a}^*(\beta, \sigma), \mathbf{g}^*(\beta, \sigma)), \\ & \mathcal{P}(\mathbf{a}^*, \mathbf{g}^*, \beta, \sigma) \leq \mathcal{B}. \end{aligned}$$

For this problem to admit a feasible solution, there must be a subsidy and a corresponding equilibrium that satisfy the budget constraint. A complication is that an equilibrium need not exist if the global complementarity parameter or the planner's subsidies are sufficiently large. The following result establishes the existence of equilibria under a regularity condition on the space of interventions and the complementarity parameter ρ .

Proposition 10. *Suppose the space of subsidy interventions is compact. There exists a parameter $\tilde{\rho} > 0$ such that if the complementarity parameter $\rho \leq \tilde{\rho}$, then every subsidy admits a pure Nash equilibrium.*

The restriction on the global complementarity parameter plays the same role as the spectral radius restriction in standard linear-quadratic network games (e.g., Ballester et al. [2006]). The additional difficulty here is that the network itself is an equilibrium object, so the strength of complementarities is jointly determined with actions. The planner's problem is then feasible when the budget is at least the equilibrium payment for one such intervention.

We would like to note that our result does not rely on the existence of equilibria for every possible intervention. When equilibria do exist our selection can be arbitrary for now, but we will impose a bit of structure below. Before turning to our analysis, we discuss several aspects of the setup.

3.2.1 Remarks on the model

Before characterizing the optimal subsidy scheme, we outline and motivate the key modeling choices. Each agent selects a non-negative action a_i and chooses link intensities g_{ij} . We briefly relate these two objects to the applications in the introduction. In the context of R&D collaborations, the action a_i represents a company's investment in research infrastructure —

such as laboratories, equipment, and researchers — while g_{ij} reflects the resources devoted to sustaining a successful collaboration with another company. In a different context of students in a social network of peers, the action denotes the student's investment in studying while the link reflects the effort expended in building and maintaining friendships within the network.

We note that it is not important for our analysis whether the intrinsic value s_{ij} of links is a coefficient on the link G_{ij} , the link intensity g_{ij} , or some combination of the two (as the difference is a pure externality term that does not change equilibrium).

We interpret the subsidies (β, σ) as monetary payments to agents, and this lets us compare action and link subsidies in common monetary units. We assume that subsidies target bilateral links G_{ij} and not agents' link intensities g_{ij} , e.g., because the planner observes the strength of the link but not who is investing to sustain the link. Notably, we do not restrict the link subsidy to be symmetric across a pair of agents. The planner can target agents differently because of heterogeneity in their action and link characteristics.

Once the planner selects a subsidy (β, σ) , agents choose actions $\mathbf{a}^*$ and link intensities $\mathbf{g}^*$ simultaneously. Studying a model with real-valued links and simultaneous link and action choices lets us study Nash equilibrium. We can therefore analyze optimal interventions via a first-order approach. Alternative solution concepts commonly used in the strategic link-formation literature, such as pairwise stability [Jackson and Wolinsky, 1996], would necessitate different methods.

3.3 Spillover Effects Of Policy Interventions

This section illustrates the key forces that determine the structure of an optimal subsidy policy. We begin by characterizing equilibrium under a given subsidy scheme. We then analyze how perturbations to action and link subsidies affect equilibrium welfare, when a first-order approach is applicable. Finally, we briefly discuss the corresponding changes in planner payments. These welfare and payment effects jointly determine the optimal allocation of the planner's limited budget across action and link subsidies.

Proposition 11. *Consider an intervention (β, σ) and a pure strategy Nash equilibrium*

$(\mathbf{a}^*, \mathbf{g}^*)$ under this intervention. Then actions satisfy

$$(\mathbf{I} - \rho \mathbf{G}^*) \mathbf{a}^* = \mathbf{b} + \boldsymbol{\beta} \quad (3.1)$$

Furthermore, any link intensity g_{ij}^* for which the equilibrium first-order condition binds satisfies

$$g_{ij}^* = \frac{\rho a_i^* a_j^* + s_{ij} + \sigma_{ij}}{f_{ij}}. \quad (3.2)$$

Given equilibrium link choices, the characterization of equilibrium actions in (3.1) matches Ballester, Calvó Armengol, and Zenou [2006]. The main difference, of course, is that links are chosen endogenously in our model, and (3.2) describes equilibrium link intensities. We will now use this characterization to compute the welfare effects of perturbing a subsidy.

Our main insight is that closely related spillover dynamics govern the effects of changing action subsidies and changing link subsidies, despite these subsidies targeting different objects.

Lemma 3. *Suppose the equilibrium selection $(\mathbf{a}^*(\boldsymbol{\beta}, \boldsymbol{\sigma}), \mathbf{g}^*(\boldsymbol{\beta}, \boldsymbol{\sigma}))$ is differentiable in a neighborhood of $(\boldsymbol{\beta}, \boldsymbol{\sigma})$. If $g_{ij}^* > 0$, then the derivatives of equilibrium welfare in β_i , β_j , and σ_{ij} are related by*

$$\frac{d\mathcal{W}}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} \left(\frac{d\mathcal{W}}{d\beta_i} a_j^* + \frac{d\mathcal{W}}{d\beta_j} a_i^* \right).$$

We now provide some intuition for this result. For ease of exposition, suppose the complementarity parameter $\rho = 1$ and curvature of cost $f_{ij} = 1$. These normalizations do not change the qualitative logic. In the general case, the marginal effect of a link subsidy is scaled by ρ/f_{ij} .

First suppose the planner marginally increases the subsidy for i 's action. What is the effect on welfare? The direct effect is that agent i responds by increasing his action a_i . This then has various spillovers: agent i now increases his link intensities with other agents g_{ij} , and other agents increase their actions a_j and the intensities g_{ji} of their links to i . Each

of these changes also have spillovers, etc. The derivative $\frac{d\mathcal{W}}{d\beta_i}$ captures the overall change in equilibrium welfare due to these cascading effects.

Now suppose the planner instead increases the subsidy for i 's link to j . Then agent i invests more in his bilateral link intensity to j . Since the overall link between i and j is stronger, they increase their actions a_i and a_j . These in turn have spillover effects as in the previous paragraph, but the key point is that these spillovers induce the same total change in welfare. So we can write $\frac{d\mathcal{W}}{d\sigma_{ij}}$ as a linear combination of $\frac{d\mathcal{W}}{d\beta_i}$ and $\frac{d\mathcal{W}}{d\beta_j}$. This basic insight also appears in Li [2023] where a planner can intervene to change exogenous links, and the lemma points out that it extends to our model with endogenous links.

The result above establishes a decomposition of link subsidy welfare effects into action subsidy effects. This plays an important role in the characterization of optimal subsidies which we discuss in the next paragraph. Before that, we highlight a key additional component governing optimality: the marginal effect on planner payments. To see the role of planner payments, consider a planner marginally increasing a link subsidy. When the actions along the endpoints of that link are large, Lemma 3 implies the welfare effect of the link subsidy can be much larger than that of an action subsidy. However, the marginal increase in payment is also large because the planner pays for the existing link level, in addition to any induced increase in equilibrium payments. This highlights an inherent trade-off which the planner needs to take into account when pinning down optimal subsidies.

We finish this section by briefly discussing how Lemma 3 is useful for our analysis. A first-order approach will compare the marginal effect — on welfare and planner payments — of a perturbation of the action subsidy with a link subsidy. A priori, there is no reason that this effect be analytically comparable as actions and links are different objects. Because of Lemma 3, the first-order condition of an action subsidy can be used to explicitly write the analogous condition for a link subsidy entirely in terms of the endpoints' actions, subsidies and primitives of the environment. This permits a simple qualitative characterization of an optimal subsidy as we will see in the following section.

3.4 Optimal Interventions With Endogenous Spillovers

This section asks whether the optimal subsidy targets actions or links. Our main result identifies a region where it is optimal to focus only on subsidizing actions and a complementary region where including link subsidies is optimal. We then discuss implications of the result in the R&D and education settings.

3.4.1 Characterization of optimal subsidies

Our analysis relies on two regularity conditions at the optimal subsidy scheme. First, we assume that the equilibrium strategy is differentiable with respect to the subsidy parameters. Because the endogenous link formation model may admit multiple equilibria for a given subsidy scheme, this assumption requires that, at the optimum, the selected equilibrium varies smoothly with local perturbations in subsidies. Second, we assume that the planner's budget constraint binds. This condition rules out perverse cases in which an increase in subsidies induces all agents to reduce both their actions and link intensities. Intuitively, these perverse cases correspond to unstable equilibria.

Assumption 2. *At the optimal subsidy (β^*, σ^*) , the following conditions hold:*

- (a) *the equilibrium selection $(\mathbf{a}^*(\beta, \sigma), \mathbf{G}^*(\beta, \sigma))$ is differentiable in a neighborhood of (β^*, σ^*) , and,*
- (b) *the planner's budget constraint binds.*

Under these conditions, we present our main results below. The qualitative structure of an intervention is governed by the baseline incentive s_{ij} to form a link. Recall, this baseline incentive is the marginal value of increasing G_{ij} from zero if all agents' actions are zero.

Theorem 4. *Let $\mathcal{W}(\mathbf{a})$ be any arbitrary planner objective and let (β^*, σ^*) denote an optimal subsidy satisfying [Assumption 2](#). If $s_{ij} \geq 0$, the optimal intervention entails no link subsidies: $\sigma_{ij}^* = \sigma_{ji}^* = 0$.*

The result says that when agents have a positive baseline incentive to form links ($s_{ij} \geq 0$), the planner does not subsidize link formation and focuses on action subsidies. This holds

despite various sources of underinvestment in links, including agents' incentives to free-ride on bilateral links.

One might wonder whether the conclusion above is driven by the fact that welfare depends only on actions. We next show that this is not the case and identify sufficient conditions under which link subsidies are necessary at the optimum.

Proposition 12. *Suppose the conditions of [Theorem 4](#) are satisfied. Consider any pair of agents i and j such that $s_{ij} < 0$ and $g_{ij}^*, g_{ji}^* > 0$. If $\beta_i^* > 0$ and $\beta_j^* > 0$, then $\sigma_{ij}^* + \sigma_{ji}^* > 0$.*

If the baseline incentive to form a link is negative ($s_{ij} < 0$), then the planner does not subsidize the actions of two connected agents i and j without also subsidizing their link.

Crucially, both these comparisons do not depend on parameters that may seem more relevant. Of particular note, the result is independent of the coefficients governing the cost of link formation, f_{ij} . For instance, f_{ij} could be much smaller than one — the curvature of the cost of actions — and even then the optimal policy does not subsidize link formation when $s_{ij} \geq 0$. Moreover, the characterization is also robust to the choice of welfare objective.

We next describe the underlying intuition for these results. Although perturbing a link subsidy can generate substantially larger welfare effects than perturbing an action subsidy (see [Lemma 3](#)), when baseline incentives are non-negative ($s_{ij} \geq 0$), subsidizing actions is a strictly cheaper way of inducing proportional spillovers in equilibrium (at the optimal subsidy). In contrast, when baseline incentives are negative ($s_{ij} < 0$), agents underinvest in link formation, and the planner optimally offsets this distortion by subsidizing links.

To formalize this, recall when agents play strategy profile $(\mathbf{a}^*, \mathbf{g}^*)$, the total payments made by the planner are:

$$\mathcal{P} = \sum_{i \in \mathcal{N}} \beta_i a_i^* + \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} \sigma_{ij} G_{ij}^*.$$

If the planner marginally increases the subsidy β_i , there are two costs: the cost a_i^* of the new subsidy and the increased costs of current subsidies as equilibrium changes in response to i 's increased effort. If the planner marginally increases the subsidy σ_{ij} , there are three

costs: the cost G_{ij}^* of the new subsidy, the subsidy cost $\frac{\sigma_{ij}}{f_{ij}}$ of i 's immediate change in link intensity g_{ij}^* , and the increased cost of current subsidies as equilibrium changes in response to i 's increased effort.

When $s_{ij} \geq 0$, the presence of this additional term implies that marginally subsidizing link formation is strictly more costly than subsidizing actions, making action subsidies the preferred instrument. In contrast, when $s_{ij} < 0$, equilibrium link intensities under zero link subsidies are small. In this case, the planner optimally exploits the low marginal cost of inducing link formation by subsidizing links.

[Proposition 12](#) provides conditions for when the planner focuses on link subsidies even though actions are the welfare-relevant quantity. We further highlight the role of joint action and link subsidies by numerically analyzing a welfare function purely determined by links. The optimal intervention can include action subsidies even when all b_i and s_{ij} are positive. So the optimal policy does not merely subsidize the type of effort valued by the planner; rather, [Theorem 4](#) and [Proposition 12](#) captures meaningful differences between links and actions.

Example 3. Suppose the welfare function

$$\mathcal{W}(\mathbf{G}) = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} G_{ij}$$

is the sum of link weights.

We consider a planner budget $\mathcal{B} = 0.01$. Individual incentives to take actions and form links are uniformly sampled as

$$\begin{aligned} b_i &\sim U([0, \bar{b}]) \quad \text{for every } i \in \mathcal{N}, \\ \text{and } s_{ij} &\sim U([0, \bar{s}]) \quad \text{for every pair } i, j \in \mathcal{N}, \end{aligned}$$

with $\bar{b} = \bar{s} = 0.01$. The cost of link formation $f_{ij} = 10$ for every pair i and j , and the cost of action $c_i = 10$ for every agent i . Finally, we set the parameter $\rho = 1$ controlling the strength of complementarities. We numerically solve for the optimal subsidy in this environment, selecting the equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ reached by starting with $(\mathbf{a}, \mathbf{g}) = (\mathbf{0}, \mathbf{0})$ and iteratively

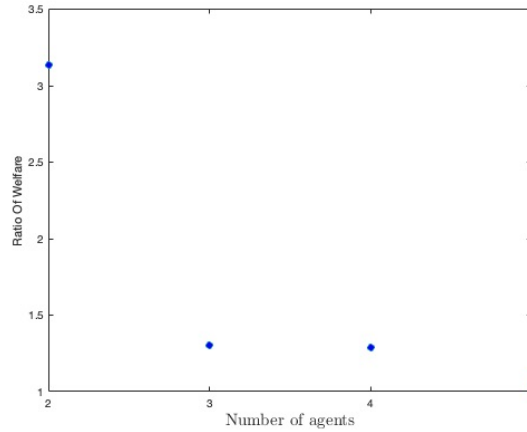


Figure 3.1: Ratio of welfare of the optimal subsidy to the welfare of the optimal subsidy which only subsidizes links as a function of the number of agents.

taking best responses. [Figure 3.1](#) illustrates the role of subsidizing actions in an optimal intervention.

The figure plots the ratio of welfare under the optimal subsidy to welfare under the best subsidy restricted to links only. A ratio exceeding one indicates that the optimal intervention necessarily includes action subsidies. We see that only subsidizing links can be far from optimal. This reinforces the role of action subsidies: they generate spillovers at a lower cost than link subsidies when link formation is endogenous.

3.4.2 Discussion of Applications

This section provides an interpretation of our result in two applications: R&D collaboration networks and peer interactions in a social network.

R&D Collaboration Networks

Governments and other policy-makers often intervene to promote innovation, and we can ask what our results imply about optimal policies.

As discussed in [Section 3.2.1](#), we interpret the action a_i as representing the firm's monetary investment in research infrastructure — such as laboratories, equipment and related capital

— while g_{ij} reflects the time and other resources devoted by firm i to sustaining a successful collaboration with another firm j .

Suppose a planner wants to maximize the total of firms' investments a_i in R&D. We argue that baseline incentives s_{ij} are likely negative in this setting: without the R&D benefits from collaborations, these links would be costly to maintain and have limited benefits. In this case, [Proposition 12](#) implies that optimal policies typically include encouraging collaborations. Whenever the planner directly subsidizes two firms and these firms collaborate, it is optimal to subsidize the collaboration as well. We note, however, that if there are large benefits to links in areas other than innovation, such as from sharing business practices, then the optimal intervention could have a different structure.

Peer Effects in Education

Interactions among peers in a network generate externalities. A substantial body of research has developed models to estimate and analyze these peer effects, particularly in the context of settings. While early models assume students peer effects come from a peer group such as a classroom (e.g., [Manski \[1993\]](#)), a large literature analyzes peer effects from friends in a social network (e.g., [Bramoullé et al. \[2009\]](#) and [Calvó-Armengol et al. \[2009\]](#)). Much of this work treats links as exogenous, but links may be endogenous in practice [[Goldsmith-Pinkham and Imbens, 2013](#)].

An important motivation for estimating peer effects is to inform the design of interventions (see, for example, [Moffitt, 2001](#) and [Viviano, 2024](#)). [Boucher, Rendall, Ushchev, and Zenou \[2024\]](#) emphasize that identifying the correct model of peer effects is crucial for designing optimal interventions. Our result sheds some light on optimal intervention policies in education, given peer effects.

As discussed in [Section 3.2.1](#), we interpret actions and links as follows: the action a_i represents a student's study effort or academic performance while the links G_{ij} capture the strength of friendships. We posit that $s_{ij} \geq 0$ in education settings: friendships have substantial appeal beyond academic benefits, so most students are likely willing to exert at least some social effort even in the absence of those benefits. Suppose a planner's objective depends on students' actions (and not on the links they form). This might be reasonable, for

example, if actions represent academic performance and peer effects reduce the effort cost of achieving a given level of performance.

Under this interpretation, [Theorem 4](#) implies that interventions targeting individual actions dominate interventions targeting links. In an education setting, student-level attempts to encourage better performance are more cost-effective than spending on connecting students. An important caveat is that we assume the planner’s objective is academic performance; if the planner values some or all links for their own sake, e.g., to decrease segregation, then the optimal policy can change (as in [Example 3](#)).

3.5 Optimal Interventions In A Benchmark Model

We next compare our results on endogenous link formation in [Section 3.4](#) with a benchmark model where links are essentially exogenous: agents choose links prior to the network game and without considering its payoffs. In this benchmark, the qualitative structure of the optimal intervention is reversed. In particular, when spillovers are large — which is the case when baseline incentives to form links are large — the planner does not subsidize the actions of two connected agents i and j without also subsidizing their link. Moreover, when the agents’ actions are small which is the case when spillovers are low and the baseline action incentive is small, the planner does not subsidize link formation.

We now describe the benchmark model. The timing is different: agents first publicly choose link intensities g_{ij} and then choose actions a_i . The link intensities $\mathbf{g}$ determine a symmetric network of spillovers $G_{ij} = G_{ji} = g_{ij} + g_{ji}$. Each agent i chooses each link intensity g_{ij} to maximize

$$s_{ij}G_{ij} + \sigma_{ij}G_{ij} - \frac{f_{ij}g_{ij}^2}{2},$$

where s_{ij} is a linear benefit from links and σ_{ij} is a subsidy parameter chosen by the planner. The linear benefit from links s_{ij} and the curvature of link intensity costs f_{ij} is symmetric across a pair of agents: $s_{ij} = s_{ji}$ and $f_{ij} = f_{ji}$.

Importantly, this benchmark assumes agents do not consider payoffs in the subsequent network game when choosing links. This may be a reasonable model in settings where agents’ primary motivations for forming links are different from the outcome of interest to the

planner. For example, students may form network ties for primarily social reasons while policymakers may care about educational outcomes affected by those networks.

The planner can also choose action subsidies β_i , and player i 's utility in the intervention game is

$$\mathcal{U}_i(\mathbf{a}, \mathbf{g}) = (b_i + \beta_i)a_i + \rho \sum_{j \in \mathcal{N} \setminus \{i\}} G_{ij}a_i a_j - \frac{a_i^2}{2}.$$

Here b_i captures an agent's individual incentive to exert effort in the absence of subsidies.

As before, given a subsidy scheme, agents play a pure strategy Nash equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ and we fix a equilibrium selection $(\mathbf{a}^*(\boldsymbol{\beta}, \boldsymbol{\sigma}), \mathbf{g}^*(\boldsymbol{\beta}, \boldsymbol{\sigma}))$. If an equilibrium does not exist given some subsidy scheme we set $\mathcal{W}$ to $-\infty$, and as before we can rule out non-existence by taking the costs of links to be sufficiently large.

A planner chooses an intervention $(\boldsymbol{\beta}, \boldsymbol{\sigma})$ to maximize a welfare function $\mathcal{W}(\mathbf{a}^*)$, which remains differentiable and increasing in each argument, subject to a budget $\mathcal{B}$.

The main result of this section provides a qualitative characterization of optimal subsidies in the large-spillover regime. Below, we formalize this regime by providing conditions on a sequence of networks:

Assumption 3. (a) (*Uniform Primitivity*) A sequence of symmetric, non-negative matrices $\{\mathbf{G}_m\}_{m=1}^{\infty}$ are said to be uniformly primitive if there exist $L \in \mathbb{N}$ and $\delta > 0$ such that for every matrix $\mathbf{G}_m$ in the sequence,

$$\frac{\mathbf{G}_m^L}{(\mu_1(\mathbf{G}_m))^L} \geq \delta \mathbf{1}\mathbf{1}^\top$$

entrywise, where $\mu_1(\mathbf{G}_m)$ is the spectral radius of $\mathbf{G}_m$.

(b) (*Critical Spectral Radius*) The limit of the spectral radius is

$$\lim_{m \rightarrow \infty} \rho \cdot \mu_1(\mathbf{G}_m) = 1.$$

(c) *(Uniform Spectral Gap) The limit of the spectral gap*

$$\lim_{m \rightarrow \infty} \rho \cdot (\mu_1(\mathbf{G}_m) - \mu_2(\mathbf{G}_m)) > 0.$$

We provide an interpretation of each condition. Part (a) considers a sequence of environments whose associated networks are uniformly primitive (see [Jackson \[2008\]](#) for the standard definition of a primitive network). This condition of connectivity admits a natural graph-theoretic interpretation. In a uniformly primitive sequence of graphs, every pair of vertices are connected to one another by a walk whose length is bounded uniformly along the sequence. Moreover, the total weight of walks between any pair of vertices is uniformly bounded away from zero.⁵

Next, Part (b) requires the spectral radius of the sequence of networks to converge to one. At this stability boundary, small changes in any agent's incentives are propagated and amplified through the network, leading to a large equilibrium response across agents.

The following theorem establishes that in such a high spillover regime, the planner does not subsidize the actions of two connected agents without also subsidizing their link.

Theorem 5. *Consider any welfare objective $\mathcal{W}(\mathbf{a})$ and a sequence of environments with optimal subsidy profiles $\{(\boldsymbol{\beta}^*, \boldsymbol{\sigma}^*)_m\}_{m \in \mathcal{N}}$ satisfying [Assumption 2](#) and [Assumption 3](#). Then the following holds for all sufficiently large m . Consider any pair of agents i and j such that $g_{ij}^*, g_{ji}^* > 0$. If $\beta_i^* > 0$ and $\beta_j^* > 0$, then $\sigma_{ij}^* > 0$ and $\sigma_{ji}^* > 0$.*

We now provide some intuition for this result. Recall that equilibrium actions are given by

$$\mathbf{a}^* = (\mathbf{I} - \rho \mathbf{G})^{-1} (\mathbf{b} + \boldsymbol{\beta}).$$

If the network is already sufficiently connected, the returns to link subsidies can be much higher than the returns to action subsidies. To see why, as the spectral radius of the spillover

⁵Consider a weighted graph $G = (V, E)$ with vertex set V and edge set E . The weight of an edge is determined by the function $w : E \rightarrow \mathbb{R}_{\geq 0}$. A length L walk in the graph is a sequence of vertices $\{v_1, v_2, \dots, v_{L+1}\}$ such that every pair of consecutive vertices $(v_i, v_{i+1}) \in E$. The weight of a walk is given by the product of weights along each edge in the walk, that is, $\prod_{1 \leq i \leq L} w(\{v_i, v_{i+1}\})$. The total weight of walks between a pair of vertices is the sum of all possible walks between them.

matrix $\mu(\rho\mathbf{G})$ approaches 1 from below, agent actions diverge to infinity. So if the planner's budget is large enough to induce $\mu(\rho\mathbf{G})$ near 1, this can induce very high welfare (as long as the welfare function values very high actions enough). In this region, actions increase very steeply as links are strengthened. The returns to action subsidies, by contrast, grow more slowly: fixing the network, subsidies to individual effort increase actions linearly. Consequently, if the planner subsidizes actions, they must also subsidize link formation.

This result stands in sharp contrast to the structure of an optimal subsidy scheme in the endogenous network formation model (see [Theorem 4](#)). In that model, the planner does not subsidize link formation when spillovers are large as induced by non-negative s_{ij} .

Finally, we identify conditions under which the planner does not subsidize links. The following can be shown: any pair of agents i and j with $g_{ij}^*, g_{ji}^* > 0$ and whose actions satisfy $a_i^* a_j^* < s_{ij}$, have link subsidies $\sigma_{ij}^* = \sigma_{ji}^* = 0$. This observation says that when agents' actions are small the planner does not subsidize links and instead focuses on actions. Such a condition is satisfied when both the baseline link incentive s_{ij} and the baseline action incentive b_i is small. This is in contrast to [Proposition 12](#) where the planner focuses on link subsidies when links are small as induced by a negative s_{ij} .

3.6 Optimal subsidy in general environments

The analysis thus far has relied on quadratic functional form assumptions. This section asks the following question: what is the structure driving each part of our main result. We identify different environments within a general model in which the optimal subsidy scheme continues to satisfy the conclusions of [Theorem 4](#) and [Proposition 12](#).

The agents take real-valued actions $a_i \geq 0$ at a cost given by a strictly increasing and convex function $C_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. The standalone marginal benefit from taking an action is b_i which lies in the interval $[\underline{b}, \bar{b}]$ where $\bar{b} > \underline{b} > 0$. Agent i choose link intensities $g_{ij} \geq 0$ with other agents $j \in \mathcal{N} \setminus \{i\}$ at a cost given by a strictly increasing and convex function $F_{ij} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. The link intensity cost is symmetric across a pair of agents: $F_{ij}(\cdot) = F_{ji}(\cdot)$. The standalone benefit from forming a link is s_{ij} which lies in the interval $[-\bar{s}, \bar{s}]$ where $\bar{s} > 0$. Finally, the link incentive is symmetric across pairs, that is, $s_{ij} = s_{ji}$.

The link intensities g_{ij} and g_{ji} determine the strength $G_{ij} = G_{ji} = g_{ij} + g_{ji}$ of an undirected link between i and j . In addition, spillovers depend on individual actions: they are determined by individual i 's action through a strictly increasing function $u_{ij} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and by individual j 's action through a strictly increasing function $\tilde{u}_{ij} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. The payoff at a zero action is zero, that is, $u_{ij}(0) = \tilde{u}_{ij}(0) = 0$. The dependence on actions is symmetric across a given pair of agents, that is, $u_{ij}(\cdot) = \tilde{u}_{ji}(\cdot)$. The utility to agent i is

$$\begin{aligned} U_i(\mathbf{a}, \mathbf{G}) = & b_i a_i + \rho \sum_{j \in \mathcal{N} \setminus \{i\}} G_{ij} u_{ij}(a_i) \tilde{u}_{ij}(a_j) - C_i(a_i) \\ & + \sum_{j \in \mathcal{N} \setminus \{i\}} s_{ij} G_{ij} - \sum_{j \in \mathcal{N} \setminus \{i\}} F_{ij}(g_{ij}). \end{aligned}$$

If the planner chooses subsidies $(\boldsymbol{\beta}, \boldsymbol{\sigma})$, then agent i 's utility is

$$\mathcal{U}_i = U_i(\mathbf{a}, \mathbf{g}) + \beta_i a_i + \sum_{j \in \mathcal{N} \setminus \{i\}} \sigma_{ij} G_{ij}.$$

Once the planner commits to an intervention, agents play a pure strategy Nash equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$. The assumptions on equilibrium existence and equilibrium selection are identical to those in [Section 3.2](#).

The following classification of functions plays a central role in deriving conditions that extend beyond the linear-quadratic setting. We begin by introducing properties on the link formation cost function.

Definition. A function $F(\cdot)$ is

- (a) *super-quadratic*, if $GF''(G) \geq F'(G)$ for all $G \geq 0$ and $F'(G)$ is convex.
- (b) *sub-quadratic*, if $GF''(G) \leq F'(G)$ for all $G \geq 0$.

A simple example satisfying these conditions is given by the following specifications. Consider any function $F_{ij}(\cdot)$ taking the form

$$F_{ij}(g) = f_{ij} g^\gamma \quad \text{for any } \gamma \geq 2 \text{ and } f_{ij} \in \mathbb{R}_{>0}.$$

Then, $F_{ij}(\cdot)$ is super-quadratic. Furthermore, any function of the form

$$F_{ij}(g) = f_{ij}g^\gamma \quad \text{for any } \gamma \in (1, 2] \text{ and } f_{ij} \in \mathbb{R}_{>0},$$

is sub-quadratic. We next introduce properties on the function $u_{ij}(\cdot)$ that govern the role of actions in spillovers.

Definition. A function $u(\cdot)$ is

(a) *super-linear*, if $u(a) \leq au'(a)$ for all $a \geq 0$.

(b) *sub-linear*, if $u(a) \geq au'(a)$ for all $a \geq 0$.

A simple example satisfying these conditions is as follows: any function $u_{ij}(\cdot)$ taking the form

$$u_{ij}(a) = \omega_{ij}a^\kappa \quad \text{for any } \kappa \geq 1 \text{ and } \omega_{ij} \in \mathbb{R}_{>0},$$

is super-linear. Furthermore, any function of the form

$$u_{ij}(a) = \omega_{ij}a^\kappa \quad \text{for any } \kappa \in (0, 1] \text{ and } \omega_{ij} \in \mathbb{R}_{>0},$$

is sub-linear. The result below presents sufficient conditions on the environment which generalize the conclusions of [Theorem 4](#).

Theorem 6. Suppose the conditions of [Theorem 4](#) are satisfied.

(i) Suppose $s_{ij} \geq 0$. If

(a) the link formation cost functions $F_{ij}(\cdot)$ is super-quadratic, and

(b) the payoff functions $u_{ij}(\cdot)$ and $\tilde{u}_{ij}(\cdot)$ are sub-linear,

then the optimal link subsidy satisfies $\sigma_{ij}^* = \sigma_{ji}^* = 0$.

(ii) Suppose $s_{ij} < 0$. If

- (a) the link formation cost function $F_{ij}(\cdot)$ is sub-quadratic, and
 - (b) the payoff functions $u_{ij}(\cdot)$ and $\tilde{u}_{ij}(\cdot)$ are super-linear, and
 - (c) the optimal action subsidies satisfy $\beta_i^*, \beta_j^* > 0$, and the link intensities $g_{ij}^*, g_{ji}^* > 0$,
- then the optimal link subsidy satisfies $\sigma_{ij}^* + \sigma_{ji}^* > 0$.

The result generalizes parts (i) and (ii) of [Theorem 4](#) to broader classes of functional forms. Unlike in the quadratic case, the two parts of the theorem are no longer complementary (as (i) and (ii) require different assumptions on curvature). But each part clarifies the underlying forces, which we now describe.

Begin with the case of a positive baseline incentive to form links ($s_{ij} \geq 0$). The characterization above implies that when link formation costs grow sufficiently steeply and spillovers from agents' actions grow sufficiently slowly, the optimal subsidy does not subsidize link formation. The intuition is that, under steep costs, link intensities respond weakly to marginal changes in link subsidies. At the same time, slowly growing spillovers dampen the return from strengthening existing links. Accordingly, the planner focuses on action subsidies. A surprising implication of the result is that the functional form of the costs of actions can be an arbitrary convex function. In particular, the cost of actions can be much steeper than the cost of link formation.

Next, suppose agents have a negative baseline incentive to form links ($s_{ij} < 0$). When link formation costs do not grow too quickly and action complementarities grow steeply enough, the planner will not subsidize two connected agents' actions without subsidizing links. A similar intuition applies to this regime. When $s_{ij} < 0$, agents underinvest in link formation. With link formation costs that are not too steep, link intensities are more responsive to marginal changes in link subsidies. Furthermore, convex spillovers propagate the return from strengthening existing links.

3.7 Conclusion

We have considered a planner intervening in a network game with endogenous links, and compared subsidies to actions and links. When baseline incentives to form links are non-negative, the optimal intervention only subsidizes actions. The benefits to links can be very high, but it is most cost effective to subsidize actions and let agents form links. When baseline incentives to form links are negative, we give sufficient conditions for the optimal intervention to include link subsidies. Both these conclusions continue to hold for a wider non-parametric class of utility functions.

We find that the structure of interventions is reversed when links are exogenous. In particular, optimal interventions include link subsidies when spillovers are large as induced by a non-negative baseline incentive to form links. On the other hand, when link incentives and action incentives are small, the planner focuses on actions.

We view the contribution of this paper as being twofold: first, highlighting the primitives of the environment that govern the qualitative properties of optimal subsidies and second, demonstrating the structural difference in subsidies between endogenous and exogenous links. Consequently, a policymaker concerned with intervening in an economic system with peer effects will need to carefully tailor her intervention to the type of links formed.

Chapter 4

Robust Analysis of Winner-Pays-Bid Auctions With Large Supply

4.1 Introduction

Many economic environments involve the sale of multiple identical goods. A leading example is online marketplaces, where firms sell products from large inventories. An important question for policymakers is what welfare guarantees standard market mechanisms provide as markets grow large. We study this question in a canonical first-price auction as the number of available goods becomes large.

Each agent has an independent private value for obtaining a good and submits a single bid. A first-price auction allocates goods to the highest bidders, up to the total supply, and the winners pay their bids. This auction format, and variants of it, are commonly used in practice because they are simple and transparent. For instance, in 2019, Google Ad Manager transitioned to “unified first-price auctions” as it provided an increase in transparency.¹ A key feature of this mechanism, however, is that agents strategically shade their bids below their valuations. This causes inefficiencies in allocation: a high-value agent may lose out if she

¹See <https://blog.google/products/admanager/rolling-out-first-price-auctions-google-ad-manager-partners/>.

bids less aggressively than a low-value competitor. What is the extent of these inefficiencies as supply becomes large?

We perform a robust analysis of the first-price auction in a large supply setting. Our analysis is robust along two margins. First, we characterize performance by the worst-case loss in efficiency across all possible equilibria. We measure the loss in efficiency by the ratio of expected welfare in equilibrium to the expected optimal welfare. Second, we analyze performance for ϵ -approximate Bayes-Nash equilibria. A large body of work performing an asymptotic analysis assumes agents play a pure Bayes-Nash equilibrium (see for example [Olszewski and Siegel \[2016\]](#), [Fibich and Gaviols \[2010\]](#) and [Swinkels, 1999](#)), which requires agents to best-respond optimally to others' actions. This may be a strong assumption in practice. Bidders rarely have precise knowledge of others' strategies and even if they do, exact optimization of bids is challenging. Approximate equilibria are robust to some of these considerations. The main challenge is that it is analytically intractable to characterize Bayes-Nash and approximate equilibria. Furthermore, Bayes-Nash and approximate equilibria typically behave very differently and so arguments that work for one may not transfer readily to the other.

The first contribution of this paper is to establish bounds on the asymptotic loss in efficiency for ϵ -approximate equilibria. The primary object in our analysis is the k -th order statistic of equilibrium bids where k is the supply. This object characterizes the minimum price at which goods sell. It is also a signal of the level of competition in the auction. In particular, it acts as the critical bid — the bid an agent must beat to win a good — for a new entrant agent in the auction. Competition is high when the critical bid takes high realizations and low when it takes low realizations. Any inefficiency in allocation is primarily driven by uncertainty in the critical bid. In [Section 4.3](#), we derive a precise measure of this uncertainty and show how it translates into inefficiencies in the limit. This measure is the area below the distribution function of the critical bid restricted to an interval within which it always lies. When the distribution function is relatively flat over a large interval, uncertainty is high. When the distribution function is steep over a small interval, uncertainty is low. [Section 4.4](#) establishes bounds on this measure as a function of the approximation factor ϵ .

A special case of our setting is when agents play a pure Bayes-Nash equilibrium. In this

setting, under a density lower bound assumption on value distributions, inefficiencies due to strategic behavior vanish in the limit. Furthermore, this observation is independent of the rate of increase between supply and the number of participating agents: the number of agents could grow arbitrarily faster than the supply. This recovers the result of [Swinkels \[1999\]](#). Our main contribution in this setting is the characterization of the price at which *all* goods sell. Theorem 4.1 in [Swinkels \[1999\]](#) shows that a fraction of the goods, where the fraction converges to 1, sell at the market clearing price in the limit. We establish that *all* units sell at the market clearing price in the limit. A key result is that the k -th order statistic is deterministic in the limit.

When agents are approximate best-responding, a key challenge is that it is analytically intractable to characterize equilibrium distributions. We characterize the worst-case distribution function of the critical bid. This establishes an explicit bound on the measure of inefficiency as a function of the approximation factor ϵ . In particular, the measure converges to 0 as agents are arbitrarily close to best-responding.

The second contribution of this paper is to derive key structural properties of equilibria. Analyzing approximate equilibria is challenging because of discontinuities introduced by atoms in the distribution function. In [Section 4.5](#), we show that for a welfare analysis, it is without loss to analyze equilibria with no atoms. For any given approximate equilibrium with atoms, we provide an explicit construction of an approximate equilibrium that has identical welfare and, crucially, has no atoms. This reduction facilitates our asymptotic analysis. Finally, we establish continuity and monotonicity properties of distribution functions in a pure Bayes-Nash equilibrium. We believe that these structural properties may be of independent interest.

Related Work

There is a rich body of work on robust analysis in auctions; see [Roughgarden et al. \[2017\]](#) for a comprehensive survey. A central technique in this literature is the smoothness framework, introduced by [Roughgarden \[2012\]](#), [Syrgkanis and Tardos \[2013\]](#), and [Roughgarden \[2015\]](#), which has proven effective for analyzing efficiency guarantees across a range of mechanisms. [Hartline et al. \[2014\]](#) further refine this approach by decomposing smoothness into

two key components—individual efficiency and competitive efficiency—which highlight structural properties of auctions that ensure near-optimal outcomes.

We view our analysis of the k -th order statistic as a contribution to this literature. The $1 - 1/e$ bound in [Hartline et al. \[2014\]](#) was a barrier to the proof approach of separating individual efficiency and competitive efficiency. The challenge in overcoming this barrier is proving that the worst-case allocation rules for individual efficiency cannot arise in equilibrium. This paper explicitly characterizes worst-case bid distributions allowing improvements over the $1 - 1/e$ bound for both Bayes-Nash and approximate equilibria.

A broader generalization of the smoothness framework, developed by [Feldman et al. \[2016\]](#), extends the analysis of inefficiency to large games and has been applied to mechanisms such as the simultaneous uniform-price auction and greedy combinatorial auction. Their results are complementary to ours, as both address efficiency guarantees in large-scale market environments.

In the economics literature, there is a long tradition of studying large-market limits of auctions and games (see [Wilson \[1977\]](#), [Rustichini et al. \[1994\]](#), [Swinkels \[1999\]](#) and [Swinkels, 2001](#)). The study most closely related to ours is [Swinkels \[1999\]](#), who analyze large-market limits of Bayes-Nash equilibrium in multi-unit first-price auctions.

The inefficiency of finite auction environments has also received significant attention. Notable contributions include [Feldman et al. \[2013\]](#), [Bhawalkar and Roughgarden \[2011\]](#), and [Markakis and Telelis \[2015\]](#), who analyze the efficiency of uniform-price and simultaneous-item auctions. The single-unit first-price auction, in particular, has been studied extensively (see [Kaplan and Zamir, 2012](#); [Hoy et al., 2018](#)). A central challenge across this literature lies in the characterization of Bayes-Nash equilibria, which often admit complex or non-explicit forms.

In a recent breakthrough, [Jin and Lu \[2023\]](#) show that the price of anarchy for the single-unit first-price auction is $1 - 1/e^2$. Their analysis characterizes the worst-case equilibrium bid distribution by solving a differential equation. Extending this type of characterization to the multi-unit setting presents substantial analytical challenges. Nevertheless, building on their work, we establish continuity and monotonicity properties of equilibrium bid dis-

tributions—crucial elements in the application of our framework. Proving these properties requires novel inductive arguments that, to the best of our knowledge, have not previously been employed to establish similar results in the economics literature.

In orthogonal work on multi-unit mechanisms that are not winner-pays-bid, a growing body of evidence suggests that such mechanisms tend to achieve stronger welfare performance than their single-unit counterparts. For example, [Yan \[2011\]](#) uses a correlation gap argument to show that, in posted-price mechanisms with independent values, the price of anarchy converges asymptotically to 1 at a rate of $1 - O(1/\sqrt{k})$. This bound is tight when agent values are identically distributed. Similarly, [Chawla et al. \[2023\]](#) study anonymous posted-price mechanisms and demonstrate that, under independent values, the worst-case distribution of agent demand is Poisson, allowing welfare approximation ratios to be computed for any finite number of units. Building on this line of work, [Jiang et al. \[2023\]](#) establish that the exact rate of convergence of the welfare approximation ratio is $1 - \Theta(\sqrt{\ln(k)/k})$.

4.2 Model

There are $k \in \mathbb{N}$ identical units and there are n single-dimensional single-unit demand agents, indexed by $\mathcal{N} = \{1, 2, \dots, n\}$. Each agent i has a private valuation drawn independently from a distribution $\mathcal{V}_i$. We denote this auction environment by $\mathcal{A}$. The collection of value distributions satisfy the following.²

Assumption 4. (a) (*Bounded Support*) *There exists a constant $H < \infty$ such that for any k and any agent $i \in \mathcal{N}$, their value v_i is at most H .*

(b) (*Uniform Density Lower Bound*) *There exists a constant $\kappa > 0$ such that for any k and*

²Our analysis extends to the weaker *No Asymptotic Gaps* condition introduced by [Swinkels \[1999\]](#) which is stated as follows: there is a continuous function $M(\cdot)$ with $M(\delta) > 0$ for all $\delta > 0$ and such that for all $\delta > 0$, there is a k^* such that for all $k > k^*$ and for all $v \in [0, H - \delta]$

$$\frac{\sum_{i \in \mathcal{N}^{(k)}} \mathbb{P}[\mathcal{V}_i \in [v, v + \delta]]}{n} > M(\delta).$$

any agent $i \in \mathcal{N}$, for any $\epsilon > 0$ and any value $v \in [0, H - \epsilon]$, the measure of values

$$\Pr [V_i \in (v, v + \epsilon)] > \kappa \epsilon.$$

Each agent submits a single non-negative bid b_i . Units are sold using a first-price auction. The k agents with the highest bids each receive one unit and pay their submitted bid, with ties broken uniformly at random. Agents have a quasi-linear utility.

A strategy for agent i is a mapping $\mathcal{E}_i : \mathcal{V}_i \rightarrow \Delta(\mathbb{R}_{\geq 0})$ from her valuation to a distribution over bids. We will denote the distribution over bids for a value v_i by $\mathcal{E}_{i,v_i}$. Sometimes we will overload notation and denote the marginal distribution over bids by $\mathcal{E}_i$. We will clarify when we do so. We denote a collection of strategy profiles by $\mathcal{E} := (\mathcal{E}_i)_{i \in \mathcal{N}}$.

Given a strategy profile, distribution over bids determines interim outcomes. The interim allocation rule for agent i is a function $x_i : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ that specifies the probability that agent i receives a unit when submitting bid b_i . Similarly, the interim payment rule $p_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ gives agent i 's expected payment conditional on bidding b_i . The interim utility of agent i with valuation v_i and bid b_i is

$$u_i(v_i, b_i) = v_i x_i(b_i) - p_i(b_i).$$

Our solution concept is an ϵ -approximate Bayes-Nash equilibrium. We provide a definition below for completeness.

Definition (ϵ -approximate Bayes Nash Equilibrium). *A strategy profile $\mathcal{E}$ is said to be an ϵ -approximate Bayes-Nash equilibrium if for every agent $i \in \mathcal{N}$, we have*

$$\mathbb{E}_{b \sim \mathcal{E}_{i,v_i}} [u_i(v_i, b)] \geq \sup_{b_i \in \mathbb{R}_{\geq 0}} u_i(v_i, b) - \epsilon \quad \text{for every } v_i \in \text{supp}(\mathcal{V}_i).$$

We impose a mild no over-bidding assumption on equilibrium bidding behavior. Our results for pure Bayes-Nash equilibrium continue to hold without this assumption.

Assumption 5 (No Over-Bidding). *Consider any auction $\mathcal{A}$ and equilibrium $\mathcal{E}$. For any agent $i \in \mathcal{N}$ and value v , the equilibrium bid $\mathcal{E}_{i,v}$ is at most v almost surely.*

Our goal is to perform a robust analysis of equilibrium welfare as k becomes large. To that end, consider any k -unit first-price auction $\mathcal{A}^{(k)}$ and any corresponding equilibrium $\mathcal{E}^{(k)}$. Let expected equilibrium welfare be $\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})$ and expected optimal welfare be $\mathcal{O}(\mathcal{A}^{(k)})$. The rest of the paper focuses on analyzing asymptotic inefficiencies in equilibria as measured by the following quantity:

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})}.$$

4.3 Robust Framework For Equilibrium Analysis

This section introduces a framework that facilitates a robust analysis of equilibrium welfare in an auction. In general, an auction admits multiple equilibria with very different welfare. Furthermore, their structure is extremely sensitive to the agents' value distribution. This makes a direct analysis of all possible equilibria intractable. The smoothness framework provides a robust approach by establishing welfare guarantees that hold uniformly across value distributions and their corresponding equilibria.

4.3.1 Framework Parameters

We begin by defining two properties attributed to any auction: individual efficiency and competitive efficiency (see [Hartline et al. \[2023\]](#) and [Syrkanis and Tardos \[2013\]](#) for closely related definitions). Together, they determine the worst-case welfare loss in every equilibrium.

We begin with the *individual efficiency* of an auction. To introduce some notation, consider an auction $\mathcal{A}$ and any corresponding equilibrium $\mathcal{E}$. The individual efficiency of an auction defined in terms of the utility of an entrant agent who competes against the incumbents playing according to equilibrium $\mathcal{E}$. Let $\mathcal{B}^{(k)}$ denote the entrants critical bid, that is, the minimum bid required to secure allocation, and let $\hat{\mathcal{B}}$ denote its expectation. Furthermore, let $\mathcal{B}^{(k)}(\cdot)$ denote its distribution function. For example, in a multi-unit first-price auction

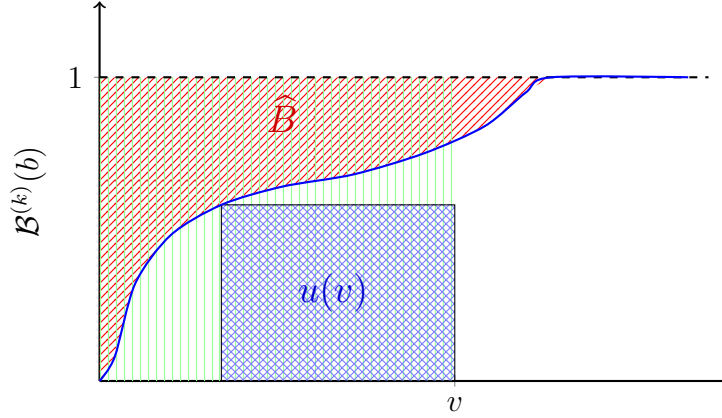


Figure 4.1: The solid blue curve plots the distribution function of the k -th highest bid $\mathcal{B}^{(k)}(\cdot)$ as a function of bids. The blue cross-hatched area is the optimal utility $u(v)$. The red area above the curve (slanting lines) is the expected critical bid $\hat{B}$. The green region (vertical lines) has area v . Individual efficiency measures what fraction of the green area does the total of blue and red area cover.

with k units, $\mathcal{B}^{(k)}$ is the k -th highest equilibrium bid. The entrant has value v best responds to obtain utility $u(v)$.

Definition (Individual Efficiency). *An auction $\mathcal{A}$ is said to be η -individually efficient if for any ϵ -approximate Bayes-Nash equilibrium $\mathcal{E}$, we have*

$$u(v) + \hat{B} \geq \eta v \quad \text{for every value } v.$$

The individual efficiency of auction $\mathcal{A}$ is the largest η such that the auction is η -individually efficient.

We provide some intuition for this definition. The individual efficiency of an auction captures an inherent tradeoff: either the optimal utility is large ($u(v)$ is large) or the agent faces strong competition in the auction (large expected critical bid $\hat{B}$). Then, the sum of utility and expected critical bid recovers a large fraction of available surplus. Figure 4.1 provides a geometric interpretation.

To further illustrate this definition, we present the individual efficiency guarantee for any multi-unit first-price auction together with a proof (see Hartline et al. [2023] for a proof and

individual efficiency guarantees in other auctions).

Example 4. The individual efficiency of any multi-unit first-price auction is at least $1 - 1/e \approx 0.63$. The proof constructs the following randomized bidding strategy for an entrant with value v . The entrant draws a bid according to the density $s(v, b) = 1/(v - b)$ supported on $b \in [0, (1 - 1/e)v]$. The incumbents' bids induce a distribution over the entrant's critical bid, denoted by $\mathcal{B}^{(k)}$. Under this strategy, the entrant's expected utility is

$$\mathbb{E}_{\hat{b} \sim \mathcal{B}^{(k)}} \left[\int_{\hat{b}}^{(1-1/e)v} (v - b) \cdot \frac{1}{v - b} db \right] = \left(1 - \frac{1}{e}\right) v - \hat{B},$$

where the expectation is over the critical bid. The optimal utility $u(v)$, then, satisfies

$$u(v) \geq \left(1 - \frac{1}{e}\right) v - \hat{B}.$$

Rearranging the equation establishes the bound on individual efficiency. Notably, the argument does not rely on the critical bid $\mathcal{B}^{(k)}$ being induced by a fixed equilibrium. As a result, the individual efficiency bound holds uniformly across all equilibria.

Next, the framework defines the *competitive efficiency* of an auction. This concept measures the extent to which competition among bidders translates into revenue for the seller. Let $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be a deterministic bid profile and $\hat{b}$ be the critical bid of an entrant agent given this bid profile. Furthermore, let $\text{Revenue}(\mathbf{b})$ denote the revenue of the auction under bid profile $\mathbf{b}$.

Definition (Competitive Efficiency). *An auction $\mathcal{A}$ is μ -competitive efficient if, for every bid profile $\mathbf{b}$ and every feasible allocation $\mathbf{y} \in \mathcal{X}$,*

$$\mu \text{Revenue}(\mathbf{b}) \geq \hat{b} \left(\sum_{i \in \mathcal{N}} y_i \right).$$

The competitive efficiency of an auction is the smallest μ such that the auction is μ -competitive efficient.

Here, competition among bidders is measured by the total critical bid of allocated agents,

given by

$$\hat{b} \sum_{i \in \mathcal{N}} y_i,$$

where $\mathbf{y}$ is a feasible allocation. Then, a small competitive efficiency (small μ) translates to high revenue for the seller, whereas a large competitive efficiency (large μ) translates to low revenue. A straightforward argument establishes the competitive efficiency for any multi-unit first-price auction.

Example 4 (continued). Any multi-unit first-price auction has competitive efficiency 1.

The inefficiency of an auction is determined by the following objects: individual efficiency, competitive efficiency and the extent to which agents are best responding to others' bids. We now provide a brief discussion on how they interact. In particular, we will analyze each of their roles in determining auction welfare and equilibrium inefficiency, interpreted as described below. Welfare can be decomposed as the sum of total agents' utility and revenue to the seller. The loss in efficiency, then, is the fraction of available surplus that welfare can recover.

Individual efficiency measures the fraction of available surplus that the auction recovers from the entrant agent. It does so by analyzing the sum of the agent's utility and the expected critical bid as a fraction of his value. Now, suppose a participating agent with identical value to the entrant is best responding. By marginally increasing the entrant player's bid, this participating agent can obtain utility that is weakly greater than the entrant. In the case the agent is ϵ -approximate best responding, the equilibrium utility is within an additive ϵ factor of the entrant. Thus, the sum of the participating agent's utility and expected critical bid recovers an identical fraction of surplus but with an additive ϵ loss.

Competitive efficiency measures what fraction of the critical bid translates into revenue for the seller. Then, to analyze the fraction of available surplus recovered by welfare, the fraction recovered by utility and the critical bid needs to be scaled by the competitive efficiency guarantee. Our focus is on multi-unit first-price auctions which have competitive efficiency 1 and so the inefficiency depends only on individual efficiency and the best response approximation. We formalize this in the result below.

Proposition 13. *Suppose the k -unit first-price auction is η -individually efficient. For every ϵ -approximate equilibrium $\mathcal{E}$, the ratio of equilibrium welfare to optimal welfare satisfies*

$$\frac{\mathcal{W}(\mathcal{E}, \mathcal{A})}{\mathcal{O}(\mathcal{A})} \geq \eta - \epsilon \cdot \frac{k}{\mathcal{O}(\mathcal{A})}$$

The result provides a useful interpretation of how individual efficiency determines welfare loss which we discuss below. This will play an important role in the asymptotic analysis of Section 4.3.2. Holding ϵ fixed, the equilibrium inefficiency is driven primarily by uncertainty in the critical bid $\mathcal{B}^{(k)}$. To see how this relates to individual efficiency η , compare the two panels in Figure 4.2. When there is uncertainty in the critical bid, as depicted in Figure 4.2(a), we have $\eta < 1$. When the critical bid is deterministic, as depicted in Figure 4.2(b), we have $\eta = 1$. Under this setting, welfare loss is only due to agents not optimally best responding.

In view of what we have just said, it is important to understand properties of the critical bid and their effect on inefficiency. This is the primary focus of the rest of the paper. In particular, we analyze the structure of $\mathcal{B}^{(k)}$ as the number of units k goes to infinity and how this governs inefficiencies in the limit. Furthermore, we study how this structure depends on agents' approximate best response behavior.

4.3.2 Framework for Asymptotic Efficiency

The framework and corresponding objects defined thus far have analyzed an auction with a fixed environment. This section analyzes the role these objects play in determining inefficiency in a sequence of auctions with an increasing number of units.

We begin with some notation and then introduce an object that plays a central role in our asymptotic analysis. Consider a sequence of k -unit first-price auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$ and a corresponding sequence of equilibria $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$. The critical bid is denoted by $\mathcal{B}^{(k)}$ and we overload notation to denote its cumulative distribution by the function $\mathcal{B}^{(k)}(\cdot)$.

A bid pair $\{\underline{b}^{(k)}, \bar{b}^{(k)}\}$, where the bids $\underline{b}^{(k)} \leq \bar{b}^{(k)}$, is defined for every auction $\mathcal{A}^{(k)}$. We now define what it means for a sequence of bid pairs to be *extremizing*.

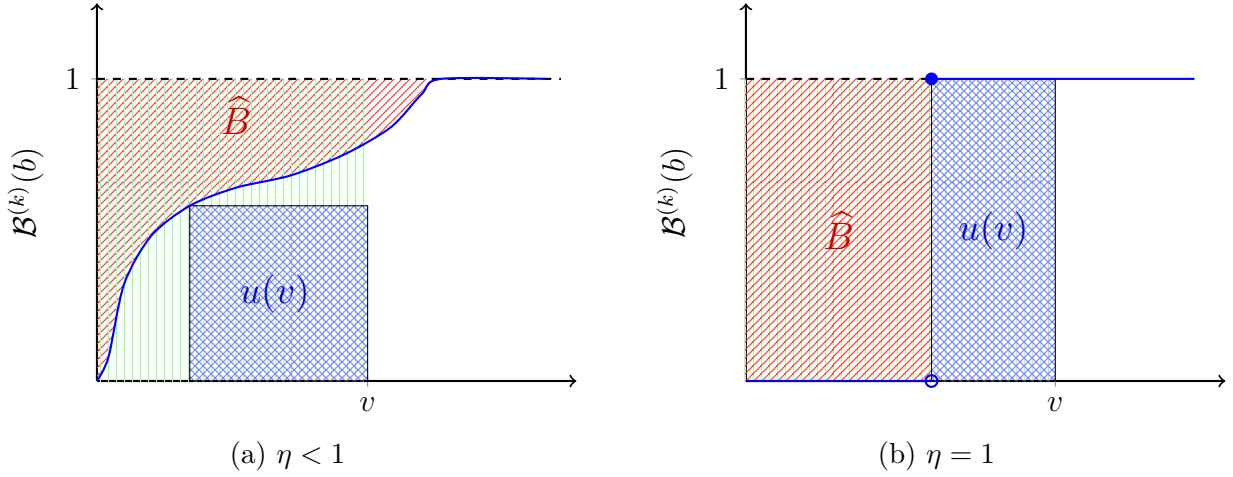


Figure 4.2: The solid blue curves plot the distribution function of the k -th highest bid. The blue cross-hatched area represents the optimal utility $u(v)$, the red area represents the expected critical bid $\hat{B}$, and the green vertically hatched region has area v .

Definition (Extremizing Sequence Of Bid Pairs). *A sequence of bid pairs*

$$\left(\left\{ \underline{b}^{(k)}, \bar{b}^{(k)} \right\} \right)_{k \in \mathbb{N}},$$

where the bids are uniformly bounded, is said to be extremizing along subsequence $(\mathcal{A}^{(k_l)})_{l=1}^{\infty}$ if

(a) The limit of the distribution function $\mathcal{B}^{(k)}$ evaluated at $\underline{b}^{(k)}$ satisfies

$$\lim_{l \rightarrow \infty} \mathcal{B}^{(k_l)} \left(\underline{b}^{(k_l)} \right) = 0.$$

(b) The limit of the distribution function $\mathcal{B}^{(k)}$ evaluated at $\bar{b}^{(k)}$ satisfies

$$\lim_{l \rightarrow \infty} \mathcal{B}^{(k_l)} \left(\bar{b}^{(k_l)} \right) = 1.$$

The relevance of this object in our analysis is the definition of the interval $\left[\underline{b}^{(k)}, \bar{b}^{(k)} \right]$. To provide some intuition, the critical bid lies within this interval almost surely in the limit. There is no uncertainty in the critical bid outside it. Any loss in efficiency due to uncertainty

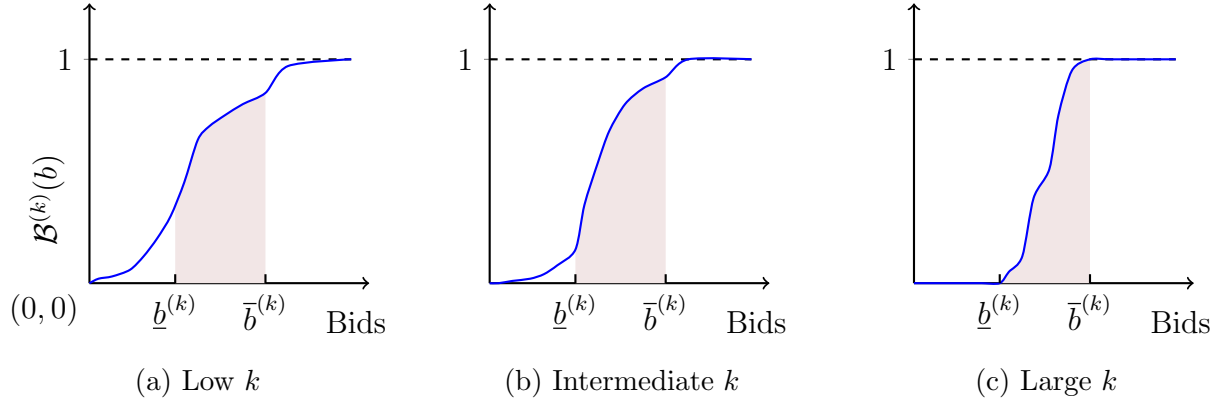


Figure 4.3: An illustration of an extremizing sequence of bid pairs. In each figure, the solid blue line plots the distribution function of the critical bid $\mathcal{B}^{(k)}$. The maroon cross-hatched region is the area below the curve between $\underline{b}^{(k)}$ and $\bar{b}^{(k)}$. Panel (a) depicts the distribution for a low number of units, Panel (b) for an intermediate number of units, and Panel (c) for a large number of units.

is, then, determined by the behavior of the critical bid within such an interval. [Figure 4.3](#) shows an illustration of an extremizing sequence of bid pairs.

When bid distributions have no atoms, a trivial extremizing bid pair sequence is one where the lower bid is zero and the higher bid is the maximum equilibrium bid. We provide a non-trivial construction in [Section 4.4](#). Finally, we present the main result of this section.

Proposition 14. *Consider any sequence of k -unit first-price auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$ satisfying [Assumption 4](#) and any corresponding sequence of $\epsilon^{(k)}$ -approximate Bayes-Nash equilibrium $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$ satisfying [Assumption 5](#). Suppose there exists a $\xi \in \mathbb{R}_{\geq 0}$ and an extremizing sequence of bid pairs $\left(\left\{\underline{b}^{(k)}, \bar{b}^{(k)}\right\}\right)_{k \in \mathbb{N}}$ along subsequence $(\mathcal{A}^{(k_l)})_{l=1}^{\infty}$ satisfying*

$$\lim_{l \rightarrow \infty} \int_{\underline{b}^{(k_l)}}^{\bar{b}^{(k_l)}} \mathcal{B}^{(k_l)}(b) db \leq \xi.$$

Then, we have

$$\lim_{l \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k_l)}, \mathcal{A}^{(k_l)})}{\mathcal{O}(\mathcal{A}^{(k_l)})} \geq 1 - \frac{\xi}{\zeta} - \frac{\epsilon}{\zeta},$$

where $\zeta > 0$ is a constant.

This result says that extent of inefficiency in the limit is determined by two factors. First, the best-response approximation ϵ . And second, the limit area below the distribution function of the critical bid and within an extremizing bid pair interval. The former is depicted by the shaded maroon area in each panel of [Figure 4.3](#). This area captures a measure of uncertainty in the critical bid relevant to limit inefficiency. The exact bound in [Proposition 14](#) follows from the geometry of individual efficiency described in [Section 4.3.1](#).

So far, we have described some key objects — critical bid and extremizing bid pair sequence — and their role in determining the extent of inefficiency in the limit. In the next section, we derive explicit bounds by characterizing the structure of these objects.

4.4 Asymptotic Welfare Analysis

We provide the main result of this section below.

Theorem 7. *Consider any sequence of k -unit first-price auctions $(\mathcal{A}^{(k)})_{k \in \mathcal{N}}$ satisfying [Assumption 4](#) and any corresponding sequence of $\epsilon^{(k)}$ -approximate Bayes-Nash equilibrium $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$ satisfying [Assumption 5](#). Let $\epsilon := \lim_{k \rightarrow \infty} \epsilon^{(k)}$.*

(a) *If $\epsilon = 0$, then*

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} = 1$$

(b) *If $\epsilon > 0$, then*

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq 1 - \zeta \epsilon \left(1 + \log \left(\frac{H}{\epsilon} \right) \right),$$

where $\zeta > 0$ depends only on the density lower bound κ .

When agents best respond optimally (or best respond optimally asymptotically), Part (a) of the result says that inefficiencies due to strategic behavior vanish in the limit. This recovers the asymptotic result of [Swinkels \[1999\]](#). Our main contribution in this setting is

the analysis, a consequence of which is the characterization of the price at which *all* units sell (see Theorem 4.1 in [Swinkels \[1999\]](#) which characterizes the selling price on *almost all* units). When agents approximate best respond with factor ϵ , Part (b) of the result establishes a bound on the loss in efficiency as a function of ϵ . Observe that the bound converges to zero as agents are arbitrarily close to best responding ($\epsilon \rightarrow 0$). We provide a proof sketch of Part (a) and Part (b) in [Section 4.4.1](#) and [Section 4.4.2](#) respectively.

Before providing details of the proof, we briefly mention a key technical contribution that facilitates our analysis. In [Section 4.5](#), we show it is without loss to analyze welfare assuming equilibrium bid distributions have no atoms. This result is surprising. Equilibrium strategies, in particular approximate equilibria, typically have atoms and this poses several analytical challenges. To that end, for any given equilibrium, we perturb the agents' strategies to construct an equilibrium having identical welfare and no atoms (see [Theorem 8](#)). For the rest of this section, we provide intuition for results assuming bid distributions have no atoms.

To establish efficiency guarantees using the robust framework described in [Section 4.3](#), we need to analyze properties of the critical bid. As discussed previously, the limit behavior of this bid governed by an extremizing sequence of bid pairs. We define two threshold bids that play a crucial role in the construction of such a sequence.

Consider any auction $\mathcal{A}^{(k)}$ and equilibrium $\mathcal{E}^{(k)}$. For every agent i , let $\mathcal{T}_i(b)$ denote the indicator random variable that takes value 1 if agent i 's equilibrium bid $\mathcal{E}_i^{(k)}$ is strictly above b and 0 otherwise. The total number of equilibrium bids exceeding b is then given by

$$\mathcal{T}^{(k)}(b) = \sum_{i \in \mathcal{N}^{(k)}} \mathcal{T}_i^{(k)}(b).$$

Definition. Fix any constant $\delta \in (0, 0.5)$. Consider the following threshold bids:

(i) The high threshold bid $\overline{\psi}_*^{(k)}$ is

$$\overline{\psi}_*^{(k)} := \min \left\{ b \geq 0 \mid \mathbb{E}[\mathcal{T}^{(k)}(b)] \leq k - k^{0.5+\delta} - 1 \right\}. \quad (4.1)$$

(ii) The low threshold bid $\underline{\psi}_*^{(k)}$ is

$$\underline{\psi}_*^{(k)} := \min \left\{ b \geq 0 \mid \mathbb{E}[\mathcal{T}^{(k)}(b)] \leq k + k^{0.5+\delta} - 1 \right\}. \quad (4.2)$$

The existence of both these quantities follows from the continuity of equilibrium bid distributions. The construction of these bids is inspired by the definition of a threshold bid for Bayes-Nash equilibrium in [Swinkels \[1999\]](#). That paper defines the high threshold bid individually for each agent and analyzes strategic behavior at this upper end. While that is sufficient to analyze efficiency, it does not characterize strategic behavior at the lower end. Our analysis provides a complete characterization of bidding behavior. In addition to this, we also show that the idea of threshold bids extends to approximate equilibria even though they behave very differently to Bayes-Nash equilibria.

We briefly argue that the bid pair sequence

$$\left(\left\{ \underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)} \right\} \right)_{k \in \mathbb{N}}$$

is extremizing (see [Lemma 15](#) for a formal result). Because bid distributions have no atoms, the inequalities in (4.1) and (4.2) hold with equality. That is, the expected total bids above $\overline{\psi}_*^{(k)}$ equals $k - k^{0.5+\delta} - 1$, and similarly, the expected total bids above $\underline{\psi}_*^{(k)}$ equals $k + k^{0.5+\delta} - 1$. A standard law of large numbers argument (see [Section D.1.1](#) for standard Chernoff bounds) argues that the k -th highest bid is within the interval $\left(\underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)} \right)$ with high probability for large k . The loss in efficiency is determined the behavior of the k -th highest bid within this interval. We will show below that this crucially depends on the agents' approximate best response factor ϵ .

4.4.1 Proof Sketch for [Theorem 7](#) Part (a)

When agents best respond optimally (or best respond optimally asymptotically), the critical bid is deterministic in the limit. In other words, the distribution function of the critical bid converges to a step function. The welfare guarantee follows from the geometry of individual efficiency in [Section 4.3.2](#). A formal result is provided below.

Proposition 15. *Consider any sequence of k -unit first-price auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$ satisfying [Assumption 4](#) and any corresponding sequence of Bayes-Nash equilibria $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$ with no atoms. Along any subsequence of auctions which admits pointwise convergence of the k -th order statistic of $\mathcal{E}^{(k)}$, the limit satisfies*

$$\lim_{k \rightarrow \infty} \mathcal{B}^{(k)}(b) = \begin{cases} 1 & \text{if } b \geq \psi^* \\ 0 & \text{if } b < \psi^* \end{cases},$$

for some $\psi^* \geq 0$.

We now provide some intuition for this result. As $k \rightarrow \infty$, suppose the following are satisfied (see [Figure 4.4](#) for an illustration of this scenario):

- threshold bid $\underline{\psi}_*^{(k)}$ does not converge to $\overline{\psi}_*^{(k)}$, and
- the area below the distribution function of the critical bid $\mathcal{B}^{(k)}$ and within the interval $\underbrace{(\underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)})}_{\mathcal{I}^{(k)}}$ is positive in the limit.

By definition of threshold bids $\underline{\psi}_*^{(k)}$ and $\overline{\psi}_*^{(k)}$, the total expected equilibrium bids in $\mathcal{I}^{(k)}$ is exactly $2k^{0.5+\delta}$. We will establish a contradiction. Any agent with value v close to $\overline{\psi}_*^{(k)}$ obtains strictly positive equilibrium utility by bidding in $\mathcal{I}^{(k)}$. This is depicted in [Figure 4.4](#) by a strictly positive shaded area below the curve of the distribution function. Thus, when $\underline{\psi}_*^{(k)}$ and $\overline{\psi}_*^{(k)}$ do not converge to each other, every agent has an constant length interval of values within $\mathcal{I}^{(k)}$ such that every value obtains strictly positive equilibrium utility in the limit (see [Lemma 17](#) for a formal result). By [Assumption 4](#), the total expected bids in $\mathcal{I}^{(k)}$ is at least κn for some constant $\kappa > 0$, which is a contradiction to the construction that the interval has at most $k^{0.5+\delta}$ bids.

The discussion above is not a formal proof. It can be that the threshold bids $\underline{\psi}_*^{(k)}$ and $\overline{\psi}_*^{(k)}$ do not converge to each other. However, the area below the distribution function must converge to zero within *some* extremizing sequence of bid pairs (see [Lemma 19](#)). [Proposition 15](#) follows from this observation.

An implication of [Proposition 15](#) is that all units sell at the same price in the limit. The

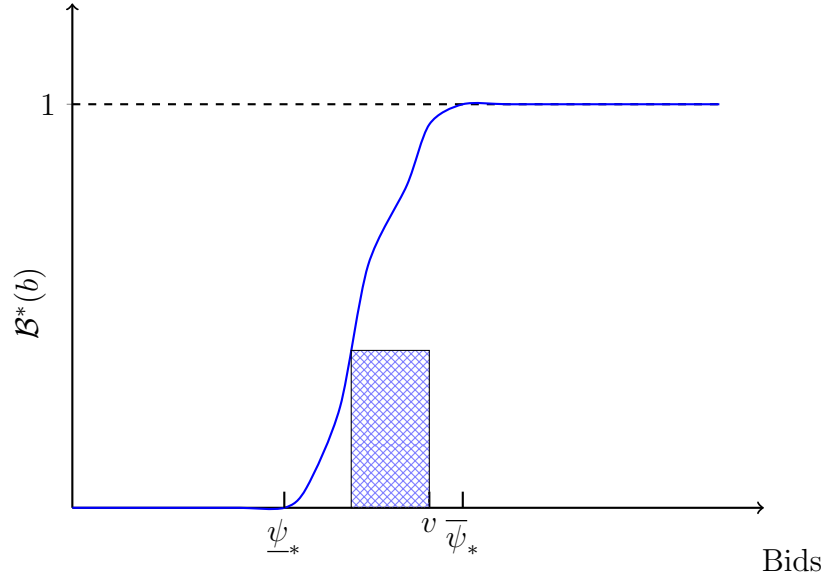


Figure 4.4: The solid curve plots the limit of the distribution function of the critical bid $\mathcal{B}^{(k)}$ as $k \rightarrow \infty$. On the x-axis, $\underline{\psi}_*$ and $\bar{\psi}_*$ denote the limit values of $\underline{\psi}_*^{(k)}$ and $\bar{\psi}_*^{(k)}$ respectively. The blue cross hatched area is a lower bound on the utility of any agent with value v .

result says that the lowest price at which a unit sells is deterministic. Combining this with the analysis in the proof of Theorem 4.1 in [Swinkels \[1999\]](#), indeed, all units sell at the same price. Furthermore, all units sell at the market clearing price: the price at which the total expected agents with a higher value equals the supply k .

Corollary 3. *All units sell at the market clearing price in the limit as $k \rightarrow \infty$.*

4.4.2 Proof Sketch for [Theorem 7 Part \(b\)](#)

When agents ϵ -approximate best respond, the contradiction argument in the previous section no longer applies to show that the critical bid is deterministic in the limit. To see why, suppose $\underline{\psi}_*^{(k)}$ and $\bar{\psi}_*^{(k)}$ do not converge to each other. As depicted in [Figure 4.4](#), a value in the interval $\mathcal{I}^{(k)}$ obtains a strictly positive optimal utility in the limit by bidding in $\mathcal{I}^{(k)}$. But agents are approximate best-responding and may bid suboptimally outside $\mathcal{I}^{(k)}$.

However, using the definition of $\underline{\psi}_*^{(k)}$ and $\bar{\psi}_*^{(k)}$, we establish a bound on the distribution function of the critical bid in $\mathcal{I}^{(k)}$ as a function of the approximation ϵ . Let $\underline{\psi}_*$ and $\bar{\psi}_*$ be

the limit of $\underline{\psi}_*^{(k)}$ and $\overline{\psi}_*^{(k)}$ respectively. We analyze two cases.

- (a) Suppose $\overline{\psi}_* - \underline{\psi}_* < \epsilon$, that is, the critical bid is concentrated in a small region (see [Figure 4.5\(a\)](#) for an illustration). The loss in efficiency measured as the area under the distribution function satisfies

$$\int_{\underline{\psi}_*}^{\overline{\psi}_*} \mathcal{B}^*(b) db < \epsilon.$$

- (b) Suppose $\overline{\psi}_* - \underline{\psi}_* > \epsilon$, that is, the critical bid is spread across a large region (see [Figure 4.5\(b\)](#) for an illustration). We show that even though the region might be large, the area under the curve remains small. By construction of $\underline{\psi}_*^{(k)}$ and $\overline{\psi}_*^{(k)}$, there must exist an agent i and a corresponding value in $[\underline{\psi}_*, \overline{\psi}_*]$ that almost surely bids in $[0, \underline{\psi}_*]$. For the purpose of illustration, suppose this value is $\overline{\psi}_*$. The equilibrium utility for this value is zero. By assumption that the agent is ϵ -approximate best-responding, the optimal utility is at most ϵ . This is only possible when the allocation probability $x_i(b)$ satisfies

$$x_i(b) \leq \frac{\epsilon}{\overline{\psi}_* - b} \quad \text{for all } b \in [\underline{\psi}_*, \overline{\psi}_* - \epsilon].$$

The bound above is precisely the iso-utility curve depicted in [Figure 4.5\(b\)](#). Under this allocation, every bid b in $[\underline{\psi}_*, \overline{\psi}_* - \epsilon]$ obtains utility exactly ϵ . The distribution function of the k -th highest bid stochastically dominates the allocation probability of every agent. Thus, the area under the distribution function is bounded as

$$\int_{\underline{\psi}_*}^{\overline{\psi}_*} \mathcal{B}^*(b) db < \epsilon \log \left(\frac{H}{\epsilon} \right) + \epsilon,$$

where H is the uniform bound on values in [Assumption 4](#). We formalize this intuition in [Lemma 18](#).

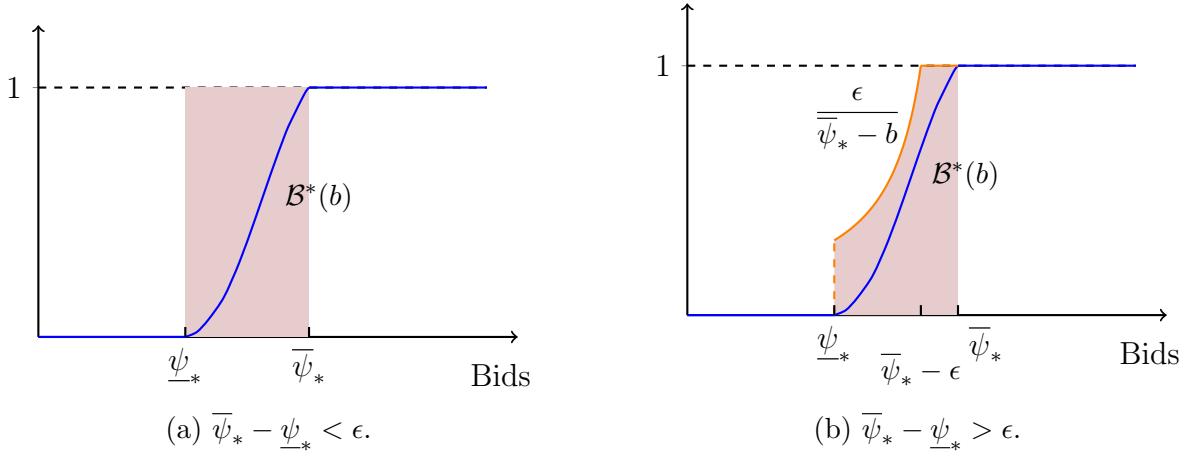


Figure 4.5: Two illustrations involving the bid distribution $\mathcal{B}^*(b)$. Panel (a) shows the bid distribution together with the maroon shaded region between $\underline{\psi}$ and $\bar{\psi}$. Panel (b) overlays the curve $\epsilon/(\bar{\psi} - b)$ on the same bid distribution.

4.5 Structural Properties Of Equilibria

This section establishes certain structural properties of equilibria that are useful not only in our analysis but, we believe, may be of independent interest. [Section 4.5.1](#) focuses on approximate Bayes-Nash equilibria while [Section 4.5.2](#) focuses on pure Bayes-Nash equilibria.

4.5.1 Continuity of approximate equilibria

Approximate equilibria are typically characterized by the presence of atoms in agents' strategies. Analyzing these objects is challenging due to discontinuities introduced by such atoms. The result below shows that for welfare analysis it is without loss to analyze equilibria with no atoms.

Theorem 8. *Consider any ϵ -approximate equilibrium $\mathcal{E}$. For every $\omega > 0$, there exists a profile of bid distributions $\tilde{\mathcal{E}}$ satisfying the following properties:*

- (a) $\tilde{\mathcal{E}}$ is an $(\epsilon + \omega)$ -approximate equilibrium.
- (b) For every agent $i \in \mathcal{N}$, the bid distribution $\tilde{\mathcal{E}}_i$ has no atoms.
- (c) $\text{welfare}(\mathcal{E}) = \text{welfare}(\tilde{\mathcal{E}})$.

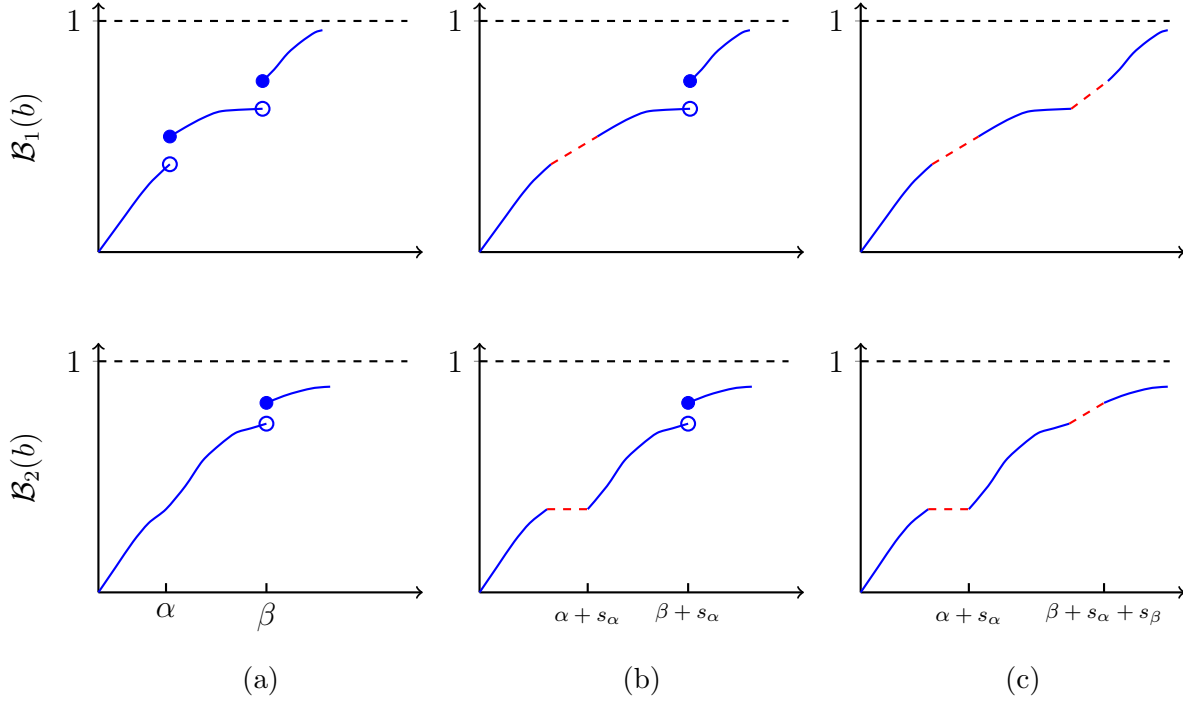


Figure 4.6: This figure illustrates the construction of an equilibrium with no atoms $\tilde{\mathcal{E}}$ beginning from $\mathcal{E}$. Column (a) depicts the distribution function of equilibrium bids for each agent: $\mathcal{B}_1(\cdot)$ for agent 1 and $\mathcal{B}_2(\cdot)$ for agent 2. Agent 1 has atoms at α and β while agent 2 only has an atom at β . Column (b) depicts the shift in bids to the right of atom α by s_α . The red dashed line shows the uniform randomization of the bid at α . Column (c) depicts the shift in bids to the right of the atom at $\beta + s_\alpha$ by s_β .

Construction of approximate equilibrium $\tilde{\mathcal{E}}$: Fix any ϵ -approximate equilibrium $\mathcal{E}$. Let $\mathcal{P}_i$ denote the location of atoms in the equilibrium bid distribution of agent i . Define $\mathcal{P} := \cup_{i \in \mathcal{N}} \mathcal{P}_i$ as the set of locations of atoms across all agents. Note that the set $\mathcal{P}$ is countable. Fix a $\omega > 0$. Since $\mathcal{P}$ is countable, for every atom $p \in \mathcal{P}$, we can assign a positive number $s_p \in \mathbb{R}_{>0}$ such that $\sum_{p \in \mathcal{P}} s_p < \omega$.

The main idea behind the construction is as follows. At every atom, add a randomization to smoothen the distribution function. An issue is that the randomization may cause bids to overlap across agents: an agent who was winning earlier may now lose. This will change agents' allocation probability and consequently welfare. To avoid this, we perform cumulative shifts of *all* agents' bids. A formal construction is provided below.

Consider the following construction of a collection of bid distributions $\tilde{\mathcal{E}} = (\tilde{\mathcal{E}}_i)_{i \in \mathcal{N}}$. For every agent $i \in \mathcal{N}$, sample the equilibrium bid $b \sim \mathcal{E}_i$. Agent i plays bid $\tilde{b}$ defined as:

$$\text{if } b \notin \mathcal{P}, \quad \text{then } \tilde{b} := b + \sum_{\{p \in \mathcal{P}: p < b\}} s_p, \quad (4.3)$$

$$\text{if } b \in \mathcal{P}, \quad \text{then } \tilde{b} \sim b + \sum_{\{p \in \mathcal{P}: p < b\}} s_p + s_b \cdot U[0, 1]. \quad (4.4)$$

Here, $U[0, 1]$ is the uniform random variable in the interval $[0, 1]$. Crucially, this variable is independent across atoms and across agents. [Figure 4.6](#) provides a pictorial construction of $\tilde{\mathcal{E}}$ in a simple two agent setting. A proof of [Theorem 8](#) is provided in [Section D.1](#).

4.5.2 Continuity and Monotonicity of Bayes-Nash equilibria

This section presents a continuity and monotonicity result for pure Bayes-Nash equilibrium. Before we present the result, we introduce some notation. Let $\gamma^{(k)}$ denote the infimum and $\lambda^{(k)}$ denote the supremum of the support of the critical bid $\mathcal{B}^{(k)}$. For instance, under [Assumption 4](#), the infimum of the critical bid's support is $\gamma^{(k)} = 0$.

Theorem 9. *Consider any k -unit first-price auction $\mathcal{A}^{(k)}$ and any corresponding pure Bayes-Nash equilibrium $\mathcal{E}^{(k)}$. The following is satisfied:*

- (a) *For every agent i , the equilibrium bid distribution $\mathcal{E}_i^{(k)}$ has no atoms in the interval $(\gamma^{(k)}, \lambda^{(k)})$. Furthermore, at most $(k - 1)$ agents have an atom at $\lambda^{(k)}$.*
- (b) *The critical bid $\mathcal{B}^{(k)}$ has no atoms in $(\gamma^{(k)}, \lambda^{(k)}]$. Moreover, its distribution function is strictly increasing in that interval.*

We use an inductive argument in conjunction with a neighborhood deviation argument to establish the result above. See [Section D.2](#) for a formal proof and additional structural properties.

4.6 Conclusion

We have performed a robust analysis of a multi-unit first-price auction with independent private values. We have also derived key continuity and monotonicity properties of approximate and Bayes-Nash equilibria.

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Appendix A

Appendix for Incentive Design With Spillovers

A.1 Omitted proofs

This section proves all results from [Section 1.2](#), [Section 1.3](#), and [Section 1.4](#). Proofs from [Section 1.5](#) are in the Online Appendix.

A.1.1 Proof of [Fact 1](#)

For contracts in this compact space, agent best responses will also be contained in a compact space. We want to show the supremum π of attainable principal contracts is attainable. Choose a sequence of contracts $\boldsymbol{\tau}^{(i)}$ giving payoffs converging to π and let $(\boldsymbol{a}^*)^{(i)}$ be the corresponding equilibria. (If π is zero, then the contract giving zero payments under all outcomes is optimal. So we can assume that π is positive and thus an equilibrium exists under $\boldsymbol{\tau}^{(i)}$ for i sufficiently large.) By compactness, we can choose a subsequence of $\boldsymbol{\tau}^{(i)}$ such that $\boldsymbol{\tau}^{(i)} \rightarrow \boldsymbol{\tau}^*$ and $(\boldsymbol{a}^*)^{(i)} \rightarrow \boldsymbol{a}^*$. By upper-hemicontinuity of equilibrium, the action profile $\boldsymbol{a}^*$ is an equilibrium under $\boldsymbol{\tau}^*$. Since the principal's payoffs are continuous in the contract and actions, the contract $\boldsymbol{\tau}^*$ attains the optimal payoff π .

A.1.2 Proof of Lemma 1

We begin by observing that under any optimal contract,

$$a_i^*(\boldsymbol{\tau}^*) = 0 \iff \tau_i^*(s) = 0, \quad \text{for all } s \in \mathcal{S}.$$

That is, at an optimal contract, an agent i exerts zero effort at equilibrium if and only if it does not receive a payment from the contract at any outcome.

To see why the observation holds, suppose at an optimal contract $\boldsymbol{\tau}^*$, there is an agent i who receives positive payment $\tau_i^*(s) > 0$ under an outcome s but chooses action $a_i^*(\boldsymbol{\tau}^*) = 0$. Then the principal receives a strictly higher profit under the contract $\boldsymbol{\tau}^\dagger$ which sets $\tau_i^\dagger(s) = 0$ for all outcomes s and is otherwise equal to $\boldsymbol{\tau}^*$. At this contract, agent i chooses action $a_i = 0$ for any profile of actions $\mathbf{a}_{-i}$ played by other agents. Thus, the equilibrium $\mathbf{a}^*(\boldsymbol{\tau}^*)$ under contract $\boldsymbol{\tau}^*$ is also an equilibrium profile under contract $\boldsymbol{\tau}^\dagger$. Since outcome s occurs with positive probability under any team performance, the principal's expected payments to agents are strictly higher under $\boldsymbol{\tau}^*$ than $\boldsymbol{\tau}^\dagger$. The other direction is straightforward.

With the observation established, we analyze the change in team performance as the transfers to agents are perturbed. Consider contract $\boldsymbol{\tau}$ and any agent i for which there exists an outcome s' such that $\tau_i(s') > 0$. For any outcome s , consider marginally increasing $\tau_i(s)$. The change induced by this perturbation is

$$\frac{\partial Y}{\partial \tau_i(s)} = \nabla Y(\mathbf{a}^*)^T \cdot \frac{\partial \mathbf{a}^*}{\partial \tau_i(s)}, \quad (\text{A.1})$$

where $\mathbf{a}^*$ is the equilibrium action profile for the contract $\boldsymbol{\tau}$. The substance of the proof is analyzing the second term on the right-hand side of (A.1).

First, consider any agent j such that $a_j^*(\boldsymbol{\tau}) = 0$. The change in their equilibrium action due to an increase in $\tau_i(s)$ is zero. The utility to agent j at contract $\boldsymbol{\tau}$ is

$$\mathcal{U}_j = \sum_{s' \in \mathcal{S}} P_{s'}(Y^*) u_j(\tau_j(s')) - C_j(a_j).$$

At contract $\boldsymbol{\tau}$, agent j receives no payment under any outcome, so he has a unique best

response of $a_j^* = 0$.

It is thus without loss to analyze the change in equilibrium actions of agents j that take a strictly positive action in profile $\mathbf{a}^*$, that is, $a_j^* > 0$. The analysis from here on focuses on such agents, overloading notation to represent the actions of these agents by $\mathbf{a}^*$.

We will show that the change in equilibrium actions $\mathbf{a}^*$ as the transfer $\tau_i(s)$ increases is

$$\frac{\partial \mathbf{a}^*}{\partial \tau_i(s)} = [\mathbf{I} - \mathbf{H}^{-1} \mathbf{U} \mathbf{G}]^{-1} \mathbf{H}^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} P'_s(Y) u'_i(\tau_i(s)) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)} [\mathbf{H} - \mathbf{U} \mathbf{G}]^{-1} \mathbf{d}, \quad (\text{A.2})$$

for some vector $\mathbf{d}$.

Consider the equilibrium action profile $\mathbf{a}^*$. For an agent j , the first-order conditions imply a_j^* must solve the equation

$$C'_j(a_j) = \left(\sum_{s' \in \mathcal{S}} P'_{s'}(Y) u_j(\tau_j(s')) \right) \frac{\partial Y}{\partial a_j}. \quad (\text{A.3})$$

To arrive at (A.2), let us implicitly differentiate (A.3) with respect to $\tau_i(s)$. The resulting expression depends on the identity of agent j in comparison to i , the agent whose payment is perturbed. For all $j \neq i$,

$$C''_j(a_j^*) \frac{\partial a_j^*}{\partial \tau_i(s)} = \left(\sum_{s' \in \mathcal{S}} P'_{s'}(Y) u_j(\tau_j(s')) \right) \left(\sum_{k=1}^n \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \tau_i(s)} \right) + \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \tau_i(s)} \cdot \sum_{s' \in \mathcal{S}} P''_{s'}(Y) u_j(\tau_j(s')). \quad (\text{A.4})$$

On the other hand, for $j = i$,

$$C_j''(a_j^*) \frac{\partial a_j^*}{\partial \tau_i(s)} = \left(\sum_{s' \in \mathcal{S}} P_{s'}'(Y) u_j(\tau_j(s')) \right) \left(\sum_{k=1}^n \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \tau_i(s)} \right) + \frac{\partial Y}{\partial a_j} P_s'(Y) u_j'(\tau_j(s)) + \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \tau_i(s)} \sum_{s' \in \mathcal{S}} P_{s'}''(Y) u_j(\tau_j(s')). \quad (\text{A.5})$$

We can combine (A.4) and (A.5) to write the resulting expression in vector form below

$$\frac{\partial \mathbf{a}^*}{\partial \tau_i(s)} = [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} P_s'(Y) u_i'(\tau_i(s)) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)} [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d},$$

where $\mathbf{d}$ is a vector with j^{th} element defined as

$$d_j := \frac{\partial Y}{\partial a_j} \cdot \sum_{s' \in \mathcal{S}} P_{s'}''(Y) u_j(\tau_j(s')).$$

The expression in (A.2) follows.

Substituting (A.2) into (A.1), the change in team performance as the transfer $\tau_i(s)$ increases is

$$\frac{\partial Y}{\partial \tau_i(s)} = \nabla Y(\mathbf{a}^*)^T [\mathbf{I} - \mathbf{H}^{-1} \mathbf{U}\mathbf{G}]^{-1} \mathbf{H}^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} P_s'(Y) u_i'(\tau_i(s)) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)} \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}.$$

Applying the definitions of α_i and c_i , we obtain

$$\frac{\partial Y}{\partial \tau_i(s)} = \alpha_i c_i P_s'(Y) \cdot \frac{u_i'(\tau_i(s))}{C_i''(a_i^*)} + \frac{\partial Y}{\partial \tau_i(s)} \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}. \quad (\text{A.6})$$

To complete the proof, we have two cases depending on whether $\nabla Y(\mathbf{a}^*)^\top [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}$ is

equal to 1. First suppose this term does not equal 1. Rearranging,

$$\frac{\partial Y}{\partial \tau_i(s)} = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}} \cdot \alpha_i c_i P'_s(Y) \cdot \frac{u'_i(\tau_i(s))}{C''_i(a_i^*)}.$$

Setting $l = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}}$ and observing l does not depend on i , we obtain the desired result.

In the remaining case $\nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d} = 1$, (A.6) simplifies to $\alpha_i c_i P'_s(Y) \cdot \frac{u'_i(\tau_i(s))}{C''_i(a_i^*)} = 0$ for any agent i and outcome s such that $\tau_i(s) > 0$. We will show in Lemma 4 that any agent that receives a payment at an optimal contract does so at an outcome where $P'_s(Y^*) > 0$. Thus $c_i = 0$ for such agents and the balance condition in Theorem 1 is satisfied trivially.

A.1.3 Proof of Theorem 1

The expected payoff for the principal under contract τ and corresponding equilibrium actions $\mathbf{a}^*$ is

$$\sum_{s' \in S} \left(v_{s'} - \sum_{i \in N} \tau_i(s') \right) P_{s'}(Y(\mathbf{a}^*)).$$

Suppose τ^* is an optimal contract inducing equilibrium $\mathbf{a}^*(\tau^*)$ with team performance Y^* . Consider outcome s and any agent i such that $\tau_i^*(s) > 0$. Then the first-order condition for $\tau_i^*(s)$ implies that

$$\underbrace{\frac{dY}{d\tau_i(s)} \sum_{s' \in S} \left(v_{s'} - \sum_{i \in N} \tau_i^*(s') \right) P_{s'}(Y^*)}_{D} = P_s(Y^*).$$

The left-hand side is the benefit from increasing $\tau_i^*(s)$ while the right-hand side is the expected additional transfer required. Since $P_s(Y^*) > 0$ by assumption, the summation labeled D is nonzero.

Substituting Lemma 1 in the above equation, we obtain

$$\begin{aligned} l\alpha_i c_i P'_s(Y^*) \cdot \frac{u'_i(\tau_i(s))}{C'''_i(a_i^*)} &= \frac{P_s(Y^*)}{D}, \\ \iff \alpha_i c_i \cdot \frac{u'_i(\tau_i(s))}{C'''_i(a_i^*)} &= \lambda_s, \end{aligned}$$

where $\lambda_s = P_s(Y^*)/(lP'_s(Y^*)D)$. Observing that λ_s is independent of i , the statement of the result follows.

A.1.4 Proof of Corollary 1

Fix any $s \in \mathcal{S}_{ij}^*$. By Theorem 1, there is a constant $\lambda_s \neq 0$ such that

$$\alpha_i c_i \cdot \frac{u'_i(\tau_i^*(s))}{C'''_i(a_i^*)} = \lambda_s, \quad \text{and} \quad \alpha_j c_j \cdot \frac{u'_j(\tau_j^*(s))}{C'''_j(a_j^*)} = \lambda_s.$$

It follows that

$$\frac{u'_i(\tau_i^*(s))}{u'_j(\tau_j^*(s))} = \frac{\alpha_j c_j}{\alpha_i c_i} \cdot \frac{C'''_i(a_i^*)}{C'''_j(a_j^*)}.$$

This is exactly the claimed equality.

A.1.5 Proof of Proposition 1

We prove a couple of lemmas which help in proving the proposition statement. The first lemma gives a condition which must hold for all outcomes at which an agent receives a positive payment.

Lemma 4. *Suppose τ^* is an optimal contract and Y^* is the induced team performance. For all s in the set of outcomes $\mathcal{S}_i^*$ where i receives a positive payment, $P'_s(Y^*) > 0$.*

Proof. Consider agent i and let $\mathcal{S}_i^*$ be the set of outcomes at which agent i receives a positive payment. If $\mathcal{S}_i^*$ is the empty set, the result holds vacuously. Otherwise, we will show that, either

$$P'_s(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*, \quad \text{or,} \quad P'_s(Y^*) < 0, \text{ for all } s \in \mathcal{S}_i^*. \quad (\text{A.7})$$

Recall from the proof of [Theorem 1](#) that

$$l\alpha_i c_i P'_s(Y^*) \cdot \frac{u'_i(\tau_i(s))}{C''_i(a_i^*)} = \frac{P_s(Y^*)}{\sum_{s' \in \mathcal{S}} (v_{s'} - \sum_{i \in N} \tau_i(s')) P'_{s'}(Y^*)}, \quad \forall s \in \mathcal{S}_i^*.$$

Since every outcome occurs with non-zero probability, it must be that $P'_s(Y^*) \neq 0$ for any outcome in $\mathcal{S}_i^*$. In the case $\mathcal{S}_i^*$ has exactly 1 outcome, it follows that $P'_s(Y^*) > 0$ or $P'_s(Y^*) < 0$ for $s \in \mathcal{S}_i^*$. Thus, suppose $\mathcal{S}_i^*$ has at least 2 outcomes. Taking the ratio of the above equation for any pair of outcomes $s_1, s_2 \in \mathcal{S}_i^*$, we obtain

$$\frac{u'_i(\tau_i^*(s_1))}{u'_i(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P_{s_2}(Y^*)} \cdot \frac{P'_{s_2}(Y^*)}{P'_{s_1}(Y^*)}.$$

Since the utility function $u_i(\cdot)$ is strictly increasing, we must have either

$$P'_s(Y^*) > 0 \text{ for } s \in \{s_1, s_2\}, \quad \text{or,} \quad P'_s(Y^*) < 0 \text{ for } s \in \{s_1, s_2\}.$$

The statement in [\(A.7\)](#) follows. We now show that $P'_s(Y^*) > 0$ for all $s \in \mathcal{S}_i^*$. The equilibrium condition for agent i is

$$C'_i(a_i^*) = \frac{\partial Y}{\partial a_i} \sum_{s \in \mathcal{S}} P'_s(Y^*) u_i(\tau_i^*(s)).$$

Since $\sum_{s \in \mathcal{S}} P'_s(Y) = 0$, the equilibrium condition can be rewritten as

$$C'_i(a_i^*) = \frac{\partial Y}{\partial a_i} \sum_{s \in \mathcal{S}_i^*} P'_s(Y^*) (u_i(\tau_i^*(s)) - u_i(0)).$$

By assumption we have a positive productivity, that is, $\frac{\partial Y}{\partial a_i} > 0$. The cost of effort is strictly increasing, that is, $C'_i(\cdot) > 0$. The utility function $u_i(\cdot)$ is strictly increasing in payments. Thus,

$$P'_s(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*$$

as desired. □

The second lemma shows the existence of a common outcome at which agents receiving a positive payment are paid.

Lemma 5. *Suppose τ^* is an optimal contract. Consider a pair of agents i and j , each with strictly concave utility functions. If there exist outcomes s_i and s_j such that $\tau_i^*(s_i) > 0$ and $\tau_j^*(s_j) > 0$, then there exists an outcome $s \in \mathcal{S}$ such that*

$$\tau_i^*(s) > 0 \text{ and } \tau_j^*(s) > 0.$$

Proof. Suppose there does not exist an outcome at which both agents receive a positive payment. Thus, the payments $\tau_i^*(s_j) = 0$ and $\tau_j^*(s_i) = 0$. The KKT first-order conditions at optimal contract τ^* are

$$lD\alpha_k c_k \cdot \frac{u'_k(\tau_k^*(s_k))}{C''_k(a_k^*)} \cdot P'_{s_k}(Y^*) - P_{s_k}(Y^*) = 0 \quad \text{for } k \in \{i, j\}. \quad (\text{A.8})$$

In addition to the above set of equations, we also have

$$lD\alpha_k c_k \cdot \frac{u'_k(0)}{C''_k(a_k^*)} \cdot P'_{s_{\{i,j\} \setminus k}}(Y^*) - P_{s_{\{i,j\} \setminus k}}(Y^*) \leq 0 \quad \text{for } k \in \{i, j\}. \quad (\text{A.9})$$

Recall $\mathcal{S}_i^*$ is the set of outcomes where agent i receives a positive payment under contract τ^* .

Since $P'_s(Y^*) > 0$ for any $s \in \mathcal{S}_i^*$ (see Lemma 4), we must have $lD\alpha_i c_i > 0$. Consider the following chain of inequalities for agent i :

$$\frac{P_{s_i}(Y^*)}{P'_{s_i}(Y^*)} < lD\alpha_i c_i \cdot \frac{u'_i(0)}{C''_i(a_i^*)} \leq \frac{P_{s_j}(Y^*)}{P'_{s_j}(Y^*)}. \quad (\text{A.10})$$

Both the inequalities follow from applying (A.8) and (A.9) to agent i . We utilize the fact that $u_i(\cdot)$ is strictly concave. We also utilize the observation that, since agent i and j receive a positive payment at outcome s_i and s_j , Lemma 4 tells us that $P'_{s_i}(Y^*) > 0$ and $P'_{s_j}(Y^*) > 0$.

Following the same computation for agent j , we obtain the inequalities

$$\frac{P_{s_j}(Y^*)}{P'_{s_j}(Y^*)} < lD\alpha_j c_j \cdot \frac{u'_j(0)}{C''_j(a_j^*)} \leq \frac{P_{s_i}(Y^*)}{P'_{s_i}(Y^*)}. \quad (\text{A.11})$$

This contradicts inequality (A.10). Thus, if two agents receive a positive payment at some (potentially different) outcomes under the optimal contract, then there must exist an outcome at which both agents receive a positive payment. $\square$

Proof of Proposition 1. Consider agents i and j with identical strictly concave utility functions $u_i(\cdot) = u_j(\cdot)$. The statement trivially holds if either agent i or agent j receives a 0 payment at all outcomes. Thus, consider a scenario where there exist outcomes s_i and s_j such that

$$\tau_i^*(s_i) > 0 \text{ and } \tau_j^*(s_j) > 0.$$

By Lemma 5, it suffices to show that when there exists an outcome such that both agents i and j receive a positive payment at this outcome, then

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S} \text{ or } \tau_j^*(s) \geq \tau_i^*(s) \text{ for all } s \in \mathcal{S}$$

(or both).

Let $\mathcal{S}_{ij}^*$ be the set of outcomes at which both agents receive a positive payment under contract τ^* . The set $\mathcal{S}_{ij}^*$ is non-empty. We can assume without loss of generality that $\tau_i^*(s) \geq \tau_j^*(s)$ for some outcome $s \in \mathcal{S}_{ij}^*$. We show that then

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S}.$$

Applying Corollary 1, we must have

$$\left| \frac{\alpha_i c_i}{C''_i(a_i^*)} \right| \geq \left| \frac{\alpha_j c_j}{C''_j(a_j^*)} \right|.$$

Additionally, $\alpha_i c_i$ and $\alpha_j c_j$ are either both positive or negative. Further applying Corollary 1

to any outcome $s' \in \mathcal{S}_{ij}^*$, the ratio of marginal utilities satisfies

$$\frac{u'_i(\tau_i^*(s'))}{u'_j(\tau_j^*(s'))} = \frac{\alpha_j c_j C''_i(a_i^*)}{\alpha_i c_i C''_j(a_j^*)} \leq 1.$$

This implies that agent i receives a weakly larger payment than agent j under all outcomes in $\mathcal{S}_{ij}^*$, that is,

$$\tau_i^*(s) \geq \tau_j^*(s) \quad \text{for all } s \in \mathcal{S}_{ij}^*.$$

We will show that this ordering on payments holds for outcomes in the set $\mathcal{S} \setminus \mathcal{S}_{ij}^*$ as well. The ordering trivially holds at outcomes where $\tau_j^*(s) = 0$. Consider an outcome s at which $\tau_j^*(s) > 0$ but $\tau_i^*(s) = 0$. We show that such an outcome cannot exist at an optimal contract τ^* . We showed in the proof of [Theorem 1](#) that the first-order condition for the principal is

$$lD\alpha_j c_j \cdot \frac{u'_j(\tau_j^*(s))}{C''_j(a_j^*)} \cdot P'_s(Y^*) - P_s(Y^*) = 0.$$

Since the utility to agent j is strictly increasing and the cost of effort is strictly convex, this implies that

$$lD\alpha_j c_j P'_s(Y^*) > 0.$$

Recall that $\left| \frac{\alpha_i c_i}{C''_i(a_i^*)} \right| \geq \left| \frac{\alpha_j c_j}{C''_j(a_j^*)} \right|$ and they are either both positive or negative. Using the fact that $u_i(\cdot)$ and $u_j(\cdot)$ are strictly increasing identical utility functions, and thus $u'_i(0) = u'_j(0) > 0$, and $P'_s(Y^*) > 0$ (see [Lemma 4](#)), we conclude

$$lD\alpha_i c_i \cdot \frac{u'_i(0)}{C''_i(a_i^*)} \cdot P'_s(Y^*) \geq lD\alpha_j c_j \cdot \frac{u'_j(0)}{C''_j(a_j^*)} \cdot P'_s(Y^*). \quad (\text{A.12})$$

Now, consider the following chain of inequalities:

$$\begin{aligned}
lD\alpha_i c_i \cdot \frac{u'_i(0)}{C''_i(a_i^*)} \cdot P'_s(Y^*) - P_s(Y^*) &\geq lD\alpha_j c_j \cdot \frac{u'_j(0)}{C''_j(a_j^*)} \cdot P'_s(Y^*) - P_s(Y^*), \\
&> lD\alpha_j c_j \cdot \frac{u'_j(\tau_j^*(s))}{C''_j(a_j^*)} \cdot P'_s(Y^*) - P_s(Y^*), \\
&= 0.
\end{aligned}$$

The first inequality follows from (A.12). The second inequality follows from the fact that $u_j(\cdot)$ is strictly concave and $lD\frac{\alpha_j c_j}{C''_j(a_j^*)}P'_s(Y^*) > 0$. The left-hand side in the above chain of inequalities is the derivative of the principal's objective in $\tau_i(s)$. The derivative being positive contradicts the optimality of τ^* , so the statement of the proposition holds. $\square$

A.1.6 Proof of Corollary 2

Recall from the proof of Theorem 1 that

$$l\alpha_i c_i P'_s(Y^*) \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)} = \frac{P_s(Y^*)}{\sum_{s' \in \mathcal{S}} (v_{s'} - \sum_{i \in N} \tau_i(s')) P'_{s'}(Y^*)}, \quad \forall s \in \mathcal{S}_i^*.$$

Taking the ratio of the above equation for any pair of outcomes $s_1, s_2 \in \mathcal{S}_i^*$, we obtain

$$\frac{u'_i(\tau_i^*(s_1))}{u'_i(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P'_{s_1}(Y^*)} \cdot \frac{P'_{s_2}(Y^*)}{P_{s_2}(Y^*)}.$$

The statement is proved.

A.1.7 Proof of Lemma 2

We begin by deriving an expression for an agent's productivity α_i . By definition,

$$\alpha_i = \frac{\partial Y}{\partial a_i} = k_i + \rho \underbrace{\left(\sum_{j \in \mathcal{N}} \beta_j a_j \right)}_{\tilde{B}_1} \beta_i. \quad (\text{A.13})$$

Because actions satisfy $a_j \geq 0$ for every j , and $\beta_j \geq 0$ by assumption, the term $\tilde{B}_1 \geq 0$.

We next derive an expression for an agent's centrality c_i . By definition, at any contract $\boldsymbol{\tau}$ and corresponding equilibrium $\boldsymbol{a}^*$, the centrality

$$\boldsymbol{c}^\top = \boldsymbol{\alpha}^\top [\boldsymbol{I} - \rho P'(Y^*) \boldsymbol{T} \boldsymbol{\beta} \boldsymbol{\beta}^\top]^{-1}, \quad (\text{A.14})$$

where $\boldsymbol{T} = \text{diag}(u_1(\tau_1), \dots, u_n(\tau_n))$. Using the Sherman-Morrison formula, the inverse matrix is

$$[\boldsymbol{I} - \rho P'(Y^*) \boldsymbol{T} \boldsymbol{\beta} \boldsymbol{\beta}^\top]^{-1} = \boldsymbol{I} + \frac{\rho P'(Y^*) \boldsymbol{T} \boldsymbol{\beta} \boldsymbol{\beta}^\top}{1 - \rho P'(Y^*) \boldsymbol{\beta}^\top \boldsymbol{T} \boldsymbol{\beta}}. \quad (\text{A.15})$$

As established in the proof of [Lemma 6](#), the spectral radius of the spillover matrix at any contract and corresponding equilibrium is strictly smaller than 1, and in the rank-one case this radius equals $\rho P'(Y^*) \boldsymbol{\beta}^\top \boldsymbol{T} \boldsymbol{\beta}$, so the denominator in (A.15) is strictly positive. Substituting (A.15) in (A.14), we get

$$\boldsymbol{c}^\top = \boldsymbol{\alpha}^\top + \underbrace{\frac{\rho P'(Y^*) \boldsymbol{\alpha}^\top \boldsymbol{T} \boldsymbol{\beta}}{1 - \rho P'(Y^*) \boldsymbol{\beta}^\top \boldsymbol{T} \boldsymbol{\beta}}}_{\tilde{B}_2} \boldsymbol{\beta}^\top,$$

where $\tilde{B}_2 \geq 0$. Substituting (A.13) in the expression above, we get

$$c_i = k_i + (\tilde{B}_1 + \tilde{B}_2) \beta_i. \quad (\text{A.16})$$

Finally, consider any optimal contract $\boldsymbol{\tau}^*$. By [Theorem 1](#), there exists a constant λ such that for any agent receiving positive pay

$$\alpha_i c_i u'_i(\tau_i) = \lambda.$$

Substituting (A.13) and (A.16) in the balance condition, it follows that there exists $B_1 \geq 0$ and $B_2 \geq 0$ such that for all agents i ,

$$u'_i(\tau_i^*) \propto (k_i^2 + B_1 k_i \beta_i + B_2 \beta_i^2)^{-1}.$$

The statement is proved.

A.1.8 Proof of Theorem 2

This proof utilizes Lemma 2, which characterizes an optimal contract for any given environment. We will also require a characterization of equilibrium action profiles under a given contract. We provide the characterization for any linear-quadratic production function without requiring rank-one structure, as the more general version will be useful for later proofs.

Lemma 6. *Consider the setting of Section 1.2.1. Fixing τ , there exists a unique pure Nash equilibrium. The equilibrium actions $\mathbf{a}^*$ and team performance Y^* solve the equations*

$$[\mathbf{I} - P'(Y^*)\mathbf{T}\mathbf{G}]\mathbf{a}^* = P'(Y^*)\mathbf{T}\mathbf{k} \text{ and } Y^* = Y(\mathbf{a}^*), \quad (\text{A.17})$$

where $\mathbf{T}$ is the diagonal matrix with entries $T_{ii} = u_i(\tau_i)$.

Proof. Fix a contract τ . Before we provide a proof, we make some useful observations. Any agent i that receives zero pay $\tau_i = 0$ must take a zero action in any equilibrium. Now, the probability of success $P(Y)$ is strictly increasing in Y and team performance $Y(\mathbf{a})$ is strictly increasing in each argument. Thus, any agent that receives strictly positive pay $\tau_i > 0$ has a strictly positive marginal returns to effort $\frac{\partial U_i}{\partial a_i} > 0$ at a zero action $a_i = 0$. To analyze equilibrium, it thus suffices to focus on agents that receive positive pay. For the rest of the argument, we overload notation and denote positive payments made to agents by τ . The proof proceeds in two steps.

Step (i): An action profile $\mathbf{a}^*$ is a pure Nash equilibrium if and only if it is a solution to the following system of equations:

$$[\mathbf{I} - P'(Y^*)\mathbf{T}\mathbf{G}]\mathbf{a}^* = P'(Y^*)\mathbf{T}\mathbf{k} \text{ and } Y^* = Y(\mathbf{a}^*). \quad (\text{A.18})$$

Proof of Step (i): Consider any pure Nash equilibrium $\mathbf{a}^*$. Because agents receive positive pay and the marginal returns at a zero action is strictly positive, we have $a_i^* > 0$ for every i . The KKT first-order condition for actions gives (A.18).

Conversely, fix $\mathbf{a}_{-i}$. Agent i 's payoff derivative has the sign of

$$u_i(\tau_i)P'(Y(a_i, \mathbf{a}_{-i})) - \frac{a_i}{k_i + \sum_{j \neq i} G_{ij}a_j + G_{ii}a_i}.$$

The first term is weakly decreasing in a_i by concavity of P and monotonicity of Y , while the fraction is strictly increasing in a_i . The displayed expression is therefore strictly decreasing in a_i . Thus, (A.18) makes a_i^* agent i 's unique global best response.

Step (ii): Fix a contract $\boldsymbol{\tau}$. There exists a unique solution to the following equations:

$$[\mathbf{I} - P'(Y^*)\mathbf{T}\mathbf{G}]\mathbf{a}^* = P'(Y^*)\mathbf{T}\mathbf{k} \text{ and } Y^* = Y(\mathbf{a}^*).$$

Proof of Step (ii): For y satisfying $P'(y)\rho(\mathbf{T}\mathbf{G}) < 1$, define

$$\mathbf{a}^*(y) = [\mathbf{I} - P'(y)\mathbf{T}\mathbf{G}]^{-1}P'(y)\mathbf{T}\mathbf{k}.$$

Solutions of the first-order conditions then correspond to solutions to

$$Y(\mathbf{a}^*(y)) = y.$$

The function $Y(\mathbf{a}^*(y))$ is strictly increasing in each coordinate of $\mathbf{a}^*(y)$. We analyze how $\mathbf{a}^*(y)$ changes as y increases. Consider the set

$$y_R := \{y : P'(y)\rho(\mathbf{T}\mathbf{G}) < 1\}.$$

Observe that because $P(\cdot)$ is concave, if $y \in y_R$ then $y + \epsilon \in y_R$ for any $\epsilon > 0$. We show that constrained to the set y_R , there exists a unique fixed point to the function $Y(\mathbf{a}^*(y))$. Each coordinate of $\mathbf{a}^*(y)$ is weakly decreasing in y since $P'(\cdot)$ is weakly decreasing (by our assumption $P(\cdot)$ is concave). So $Y(\mathbf{a}^*(y))$ is decreasing, meaning there is at most one solution to $Y(\mathbf{a}^*(y)) = y$. It remains to show a solution to this equation exists.

We claim that we can find y such that $Y(\mathbf{a}^*(y)) \geq y$ and $P'(y)\rho(\mathbf{T}\mathbf{G}) < 1$. If $P'(0)\rho(\mathbf{T}\mathbf{G}) < 1$, the claim holds with $y = 0$ since $Y(\mathbf{a}^*(0)) \geq 0$. Otherwise, define y_0 by $P'(y_0)\rho(\mathbf{T}\mathbf{G}) = 1$.

A solution to this equation exists since $P'(y)$ is continuous and converges to zero as $y \rightarrow \infty$. Then $Y(\mathbf{a}^*(y)) \rightarrow \infty$ as $y \rightarrow y_0$ from above, so we have $Y(\mathbf{a}^*(y_0 + \epsilon)) \geq y_0 + \epsilon$ for $\epsilon > 0$ sufficiently small. This completes the proof of the claim.

Since $Y(\mathbf{a}^*(y))$ is decreasing in y , we can also choose y large enough such that $y > Y(\mathbf{a}^*(y))$. Since $Y(\mathbf{a}^*(y))$ is continuous in y , by the intermediate value theorem this function has a fixed point, denoted by y^* . We conclude that there exists a unique solution to $Y(\mathbf{a}^*(y)) = y$ in the set y_R and a corresponding profile $\mathbf{a}^*$ of equilibrium actions.

It remains to show that there does not exist an equilibrium $\mathbf{a}^*$ with corresponding team performance Y^* such that $P'(Y^*)\rho(\mathbf{T}\mathbf{G}) \geq 1$. The case $\tau = 0$ is immediate as the only equilibrium is $\mathbf{a}^* = 0$. Take τ not identically zero and suppose there exists an equilibrium $\mathbf{a}^*$ such that $P'(Y^*)\rho(\mathbf{T}\mathbf{G}) \geq 1$. It must solve:

$$[\mathbf{I} - P'(Y^*)\mathbf{T}\mathbf{G}]\mathbf{a}^* = P'(Y^*)\mathbf{T}\mathbf{k} \text{ and } Y^* = Y(\mathbf{a}^*). \quad (\text{A.19})$$

By the Perron-Frobenius theorem,¹ there exists a left-eigenvector $\mathbf{v}$ of the matrix $P'(Y^*)\mathbf{T}\mathbf{G}$ such that $\mathbf{v}$ has strictly positive entries. Multiplying the LHS of (A.19) by the vector $\mathbf{v}$, we get

$$\begin{aligned} \mathbf{v}^T[\mathbf{I} - P'(Y^*)\mathbf{T}\mathbf{G}]\mathbf{a}^* &= [1 - P'(Y^*)\rho(\mathbf{T}\mathbf{G})]\mathbf{v}^T\mathbf{a}^* \\ &\leq 0, \end{aligned}$$

where the inequality follows from the assumption $P'(Y^*)\rho(\mathbf{T}\mathbf{G}) \geq 1$ and the fact that $\mathbf{a}^*$ has strictly positive elements. However, we also compute

$$\begin{aligned} \mathbf{v}^T[\mathbf{I} - P'(Y^*)\mathbf{T}\mathbf{G}]\mathbf{a}^* &= \mathbf{v}^T P'(Y^*)\mathbf{T}\mathbf{k} \text{ by (A.19)} \\ &> 0, \end{aligned}$$

where the inequality holds because the entries of $\mathbf{v}$ are all positive and the entries of τ are

¹For this argument, it is without loss to assume the matrix $\mathbf{T}\mathbf{G}$ is irreducible. If not, since $\mathbf{G}$ is symmetric and $\tau_i > 0$ for some agent i , we can rewrite $\mathbf{T}\mathbf{G}$ in a block diagonal form with irreducible blocks. Then $P'(Y^*)\rho(\mathbf{T}\mathbf{G})$ must be an eigenvalue of at least one block of the matrix $P'(Y^*)\mathbf{T}\mathbf{G}$. We can drop agents in all other blocks and apply the remainder of the argument to this block.

all non-negative and not identically zero. This is a contradiction, so there does not exist an equilibrium $\mathbf{a}^*$ with corresponding team performance Y^* such that $P'(Y^*)\rho(\mathbf{T}\mathbf{G}) \geq 1$. We conclude the equilibrium described above is the unique one. $\square$

We can now prove the theorem. Consider any optimal contract $\boldsymbol{\tau}$ and the corresponding equilibrium $\mathbf{a}^*$, which (by the previous lemma) is given by

$$\mathbf{a}^* = P'(Y^*) [\mathbf{I} - \rho P'(Y^*) \mathbf{T} \boldsymbol{\beta} \boldsymbol{\beta}^\top]^{-1} \mathbf{T} \mathbf{k}, \quad (\text{A.20})$$

where $\mathbf{T} = \text{diag}(u_1(\tau_1), \dots, u_n(\tau_n))$. Under this characterization, it can be established that the constants in Lemma 2 have the following expression:

$$B_1 = \rho P'(Y^*) \mathbf{k}^\top \mathbf{T} \boldsymbol{\beta} \cdot \frac{3 - 2\rho P'(Y^*) \boldsymbol{\beta}^\top \mathbf{T} \boldsymbol{\beta}}{(1 - \rho P'(Y^*) \boldsymbol{\beta}^\top \mathbf{T} \boldsymbol{\beta})^2}, \quad (\text{A.21})$$

$$\text{and } B_2 = (\rho P'(Y^*) \mathbf{k}^\top \mathbf{T} \boldsymbol{\beta})^2 \cdot \frac{2 - \rho P'(Y^*) \boldsymbol{\beta}^\top \mathbf{T} \boldsymbol{\beta}}{(1 - \rho P'(Y^*) \boldsymbol{\beta}^\top \mathbf{T} \boldsymbol{\beta})^3}. \quad (\text{A.22})$$

Given the above expressions, we establish Part (i) and Part (ii).

Before turning to each part of the argument, we characterize the spectral radius of the spillover matrix. Consider the sequence of environments in the theorem, and define

$$\mathbf{T}^{(m)} = \text{diag}(u_1(\tau_1^{(m)}), \dots, u_n(\tau_n^{(m)})).$$

Denote the induced team performance by $Y^{*(m)}$. The spillover matrix is

$$\mathbf{S}^{(m)} = \rho^{(m)} P'(Y^{*(m)}) \mathbf{T}^{(m)} \boldsymbol{\beta} \boldsymbol{\beta}^\top,$$

which is rank one. Its unique non-zero eigenvalue is

$$\mu^{(m)} = \rho^{(m)} P'(Y^{*(m)}) \boldsymbol{\beta}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} \quad (\text{A.23})$$

with associated eigenvector $\mathbf{T}^{(m)} \boldsymbol{\beta}$. Let $B_1^{(m)}$ and $B_2^{(m)}$ denote the expressions in (A.21) and (A.22), respectively, evaluated in environment m .

To establish Part (i), we want to prove that as $m \rightarrow \infty$ under $\mu^{(m)} \rightarrow 0$, the ratio of agent payments converges as follows:

$$\lim_{m \rightarrow \infty} \frac{u'_i(\tau_i^{(m)})}{u'_j(\tau_j^{(m)})} = \frac{k_j^2}{k_i^2}.$$

By Lemma 2, the ratio of payments for any pair of agents i and j is

$$\frac{u'_i(\tau_i^{(m)})}{u'_j(\tau_j^{(m)})} = \frac{k_j^2 + B_1^{(m)}k_j\beta_j + B_2^{(m)}\beta_j^2}{k_i^2 + B_1^{(m)}k_i\beta_i + B_2^{(m)}\beta_i^2} \quad (\text{A.24})$$

We will show that $B_1^{(m)}$ and $B_2^{(m)}$, given by (A.21) and (A.22) evaluated in environment m , converge to 0 as $m \rightarrow \infty$ under $\mu^{(m)} \rightarrow 0$. By assumption and (A.23), we have

$$\lim_{m \rightarrow \infty} \rho^{(m)} P'(Y^{*(m)}) \boldsymbol{\beta}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} = 0.$$

It remains to analyze the convergence of $\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta}$. Consider the following chain of inequalities:

$$\rho^{(m)} P'(Y^{*(m)}) \boldsymbol{\beta}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} \geq \underline{\beta} \rho^{(m)} P'(Y^{*(m)}) \mathbf{1}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} \geq (\underline{\beta}/\bar{k}) \rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta}.$$

Since $\underline{\beta}/\bar{k} > 0$ and the left-hand side converges to 0, the term $\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta}$ converges to 0. Substituting in (A.21) and (A.22), it follows that $B_1^{(m)}$ and $B_2^{(m)}$ both converge to 0. Thus,

$$\lim_{m \rightarrow \infty} \frac{u'_i(\tau_i^{(m)})}{u'_j(\tau_j^{(m)})} = \frac{k_j^2}{k_i^2}.$$

To establish Part (ii), we want to prove that as $m \rightarrow \infty$ under $\mu^{(m)} \rightarrow 1$, the ratio of agent

payments converges as follows:

$$\lim_{m \rightarrow \infty} \frac{u'_i(\tau_i^{(m)})}{u'_j(\tau_j^{(m)})} = \frac{\beta_j^2}{\beta_i^2}.$$

Rewriting the ratio of payments using (A.21) and (A.22), we have that

$$\begin{aligned} \frac{u'_i(\tau_i^{(m)})}{u'_j(\tau_j^{(m)})} &= \frac{(1 - \mu^{(m)})^3 k_j^2 + \rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} (1 - \mu^{(m)}) (3 - 2\mu^{(m)}) k_j \beta_j + (\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta})^2 (2 - \mu^{(m)}) \beta_j^2}{(1 - \mu^{(m)})^3 k_i^2 + \rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} (1 - \mu^{(m)}) (3 - 2\mu^{(m)}) k_i \beta_i + (\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta})^2 (2 - \mu^{(m)}) \beta_i^2} \end{aligned}$$

It suffices to establish that $\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta}$ does not converge to 0 and remains bounded as $m \rightarrow \infty$ under $\mu^{(m)} \rightarrow 1$. Consider the following chain of inequalities:

$$\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} \geq \underline{k} \rho^{(m)} P'(Y^{*(m)}) \mathbf{1}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} \geq \frac{\underline{k}}{\underline{\beta}} \rho^{(m)} P'(Y^{*(m)}) \boldsymbol{\beta}^\top \mathbf{T}^{(m)} \boldsymbol{\beta}.$$

Since $\underline{k}/\underline{\beta} > 0$ and $\mu^{(m)}$ converges to 1, it must be that $\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta}$ does not converge to 0. Next,

$$\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} \leq \bar{k} \rho^{(m)} P'(Y^{*(m)}) \mathbf{1}^\top \mathbf{T}^{(m)} \boldsymbol{\beta} \leq \frac{\bar{k}}{\underline{\beta}} \rho^{(m)} P'(Y^{*(m)}) \boldsymbol{\beta}^\top \mathbf{T}^{(m)} \boldsymbol{\beta}.$$

These inequalities establish that $\rho^{(m)} P'(Y^{*(m)}) \mathbf{k}^\top \mathbf{T}^{(m)} \boldsymbol{\beta}$ remains bounded. Thus,

$$\lim_{m \rightarrow \infty} \frac{u'_i(\tau_i^{(m)})}{u'_j(\tau_j^{(m)})} = \frac{\beta_j^2}{\beta_i^2}.$$

The statement is proved.

A.1.9 Proof of Proposition 2

As established in Section 1.4.2, under an optimal contract τ^* , the product $\alpha_i c_i$ is equal across all active agents.

Lemma 7. *If $\alpha_i c_i$ is constant across all active agents, then α_i is constant across all active agents.*

Proof. By Theorem 1, we have that

$$\alpha_i c_i \text{ is constant across agents } i.$$

We derive an expression for productivity α and centrality c that will be used throughout the proof. The productivity α is given by

$$\begin{aligned} \alpha &= \mathbf{1} + \mathbf{G}a^*, \\ &= \mathbf{1} + P'(Y^*)\mathbf{G}[\mathbf{I} - P'(Y^*)\mathbf{T}\mathbf{G}]^{-1}\tau \quad (\text{by Lemma 6}), \\ &= [\mathbf{I} - P'(Y^*)\mathbf{G}\mathbf{T}]^{-1}\mathbf{1}, \end{aligned}$$

where $\mathbf{1}$ denotes the vector whose entries are all equal to one. Furthermore, the centrality c is defined as

$$c = [\mathbf{I} - P'(Y^*)\mathbf{G}\mathbf{T}]^{-1}\alpha.$$

Suppose that there exist two agents $i^* \in N$ with $i^* = \arg \min_{k \in N} \alpha_k$ and $j^* \in N$ with $j^* = \arg \max_{k \in N} \alpha_k$ such that $\alpha_{i^*} < \alpha_{j^*}$.²

Then we have that, for agent i^* ,

$$\alpha_{i^*} c_{i^*} < \alpha_{i^*} \alpha_{j^*} \sum_{j \in N} [\mathbf{I} - P'(Y^*)\mathbf{G}\mathbf{T}]_{i^*j}^{-1} = (\alpha_{i^*})^2 \alpha_{j^*}, \quad (\text{A.25})$$

using the maximality of α_{j^*} among the α_j and the definitions of c_{i^*} and α_{i^*} . But we similarly

²We are grateful to Michael Ostrovsky for suggesting the argument in the next paragraph.

have that, for agent j^* ,

$$\alpha_{j^*} c_{j^*} > \alpha_{j^*} \alpha_{i^*} \sum_{i \in N} [\mathbf{I} - P'(Y^*) \mathbf{G} \mathbf{T}]_{j^* i}^{-1} = \alpha_{i^*} (\alpha_{j^*})^2. \quad (\text{A.26})$$

[Theorem 1](#) implies that $\alpha_{i^*} c_{i^*} = \alpha_{j^*} c_{j^*}$ for any two agents i^* and j^* , and so combining [\(A.25\)](#) and [\(A.26\)](#) implies

$$(\alpha_{i^*})^2 \alpha_{j^*} > \alpha_{i^*} (\alpha_{j^*})^2.$$

This contradicts our assumption $\alpha_{j^*} > \alpha_{i^*}$, so we must have α_i equal to some constant λ_1 for all i in N . $\square$

Part (b) follows because $\alpha_i = 1 + (\mathbf{G} \mathbf{a}^*)_i$, so Part (a) implies that $(\mathbf{G} \mathbf{a}^*)_i$ is constant across active agents. For Part (c), the equilibrium first-order condition gives $a_i^* = P'(Y^*) \tau_i^* \alpha_i$; since $P'(Y^*)$ and α_i are common across active agents, $\boldsymbol{\tau}^*$ is a common scalar multiple of $\mathbf{a}^*$, and Part (b) implies that $(\mathbf{G} \boldsymbol{\tau}^*)_i$ is constant across active agents.

A.1.10 Proof of [Proposition 3](#)

Suppose $\boldsymbol{\tau}^*$ is an optimal contract for network $\mathbf{G}$ with equilibrium team performance $Y^*(\mathbf{G}, \boldsymbol{\tau}^*)$. Consider a perturbed network $\tilde{\mathbf{G}}$ generated by increasing edge weight G_{ij} to $G_{ij} + \epsilon$ for some $\epsilon > 0$. We will show that contract $\boldsymbol{\tau}^*$ performs weakly better on network $\tilde{\mathbf{G}}$ than on $\mathbf{G}$. Since we will be comparing $\boldsymbol{\tau}^*$ across networks, we suppress the dependence of equilibrium team performance on the contract.

Consider contract $\boldsymbol{\tau}^*$ and network $\tilde{\mathbf{G}}$. We want to show that the equilibrium team performance $Y^*(\tilde{\mathbf{G}})$ is at least $Y^*(\mathbf{G})$. The equilibrium actions solve

$$\mathbf{a}^*(\tilde{\mathbf{G}}) = P'(Y^*(\tilde{\mathbf{G}})) \left[\mathbf{I} - P'(Y^*(\tilde{\mathbf{G}})) \mathbf{T}^* \tilde{\mathbf{G}} \right]^{-1} \boldsymbol{\tau}^*.$$

Suppose $Y^*(\tilde{\mathbf{G}}) < Y^*(\mathbf{G})$. It follows that $\mathbf{a}^*(\tilde{\mathbf{G}})$ is pointwise weakly greater than $\mathbf{a}^*(\mathbf{G})$ by the Neumann series and the nonnegativity of the spillover matrix. However, this is a

contradiction to $Y^*(\tilde{\mathbf{G}}) < Y^*(\mathbf{G})$ because

$$Y(\mathbf{a}, \mathbf{G}) = \sum_i a_i + \frac{1}{2} \sum_{i,j} G_{ij} a_i a_j.$$

Thus, we must have $Y^*(\tilde{\mathbf{G}}) \geq Y^*(\mathbf{G})$. Moreover, $1 - \sum_i \tau_i^* \geq 0$ because the optimal contract weakly dominates the zero contract. The profits to the principal under contract $\boldsymbol{\tau}^*$ are thus weakly higher on network $\tilde{\mathbf{G}}$ than on network $\mathbf{G}$:

$$\left(1 - \sum_i \tau_i^*\right) P(Y^*(\tilde{\mathbf{G}})) \geq \left(1 - \sum_i \tau_i^*\right) P(Y^*(\mathbf{G})).$$

Finally, the optimal contract for network $\tilde{\mathbf{G}}$ must deliver at least as high a payoff as contract $\boldsymbol{\tau}^*$ does on network $\tilde{\mathbf{G}}$.

A.1.11 Proof of Fact 3

We begin by providing sufficient conditions for strict best responses and stable equilibrium actions under Cobb-Douglas production, before turning to the CES specification.

Proof of Part (i). Consider any contract $\boldsymbol{\tau}$. For Cobb-Douglas production, team performance is

$$Y(\mathbf{a}) = \prod_{i=1}^n a_i^{\gamma_i}.$$

We begin by establishing strictness of best responses. The utility of an agent is

$$\mathcal{U}_i = \tau_i P(Y) - \frac{a_i^2}{2}.$$

The second derivative of $\mathcal{U}_i$ with respect to a_i is

$$\frac{\partial^2 \mathcal{U}_i}{\partial a_i^2} = \tau_i P''(Y) \left(\frac{\partial Y}{\partial a_i} \right)^2 + \tau_i P'(Y) \frac{\partial^2 Y}{\partial a_i^2} - 1. \quad (\text{A.27})$$

The second derivative of $Y(\cdot)$ with respect to a_i is $\gamma_i(\gamma_i - 1)Y/a_i^2$. Thus, (A.27) is strictly

negative when $P(\cdot)$ is concave and $\gamma_i \leq 1$.

We next establish the stability of equilibrium actions at the optimal contract. Observe that the action profile $a_i^* = 0$ for every agent $i \in \mathcal{N}$ is a Nash equilibrium for any contract τ . Consequently, for a given contract, the principal's preferred equilibrium is the one — if it exists — in which all agents exert positive effort. For any contract such that $\tau_i = 0$ for some agent $i \in \mathcal{N}$, the only equilibrium is one where every agent takes a zero action. For the remainder of the proof, we focus on contracts where every agent receives positive pay.

We show that when the factor shares also satisfy

$$\sum_{i \in \mathcal{N}} \gamma_i < 2,$$

a unique equilibrium with positive actions exists and is differentiable with respect to the contract τ . The result relies on the following characterization of equilibrium team performance: given τ , an equilibrium with positive actions exists if and only if there is a solution to

$$Y^{\frac{2 - \sum_i \gamma_i}{\sum_i \gamma_i}} \cdot \frac{1}{P'(Y)} = \prod_i (\tau_i \gamma_i)^{\frac{\gamma_i}{\sum_i \gamma_i}}. \quad (\text{A.28})$$

To see why, suppose a positive equilibrium $\mathbf{a}^*$ exists and let Y^* denote the corresponding team performance. Agents' actions must satisfy the equilibrium first-order conditions

$$a_i^* = \sqrt{\tau_i \gamma_i P'(Y^*) Y^*} \quad \text{for every } i \in \mathcal{N}.$$

Raising both sides to the power γ_i , multiplying across all agents and taking the power of that equation to $(2 / \sum_i \gamma_i)$ yields (A.28). For the converse direction, suppose there exists a solution to (A.28) which we denote by Y^* . The action profile defined by

$$a_i^* = \sqrt{\tau_i \gamma_i P'(Y^*) Y^*} \quad \text{for every } i \in \mathcal{N},$$

then constitutes a Nash equilibrium. This follows because when $\gamma_i \leq 1$, each agent's utility is strictly concave in own action, implying that any profile satisfying the first-order conditions is indeed a Nash equilibrium.

We now establish that when $\sum_i \gamma_i < 2$, (A.28) admits a unique positive solution for any given contract $\boldsymbol{\tau}$. By assumption, $Y^{\frac{2-\sum_i \gamma_i}{\sum_i \gamma_i}}$ is strictly increasing and $P'(\cdot)$ is weakly decreasing by concavity of $P(\cdot)$. Thus the left-hand side of (A.28) is strictly increasing which implies a unique positive solution exists. Moreover, because $P(\cdot)$ is smooth, the left-hand side of (A.28) is smooth, implying that the solution $Y^*(\boldsymbol{\tau})$ is differentiable with respect to the contract $\boldsymbol{\tau}$. Combining this with the uniqueness of the positive equilibrium action profile, it follows that the principal's preferred equilibrium is differentiable at every contract $\boldsymbol{\tau}$.

Proof of Part (ii). A similar analysis applies to CES production. Consider any contract $\boldsymbol{\tau}$. The team performance is

$$Y(\mathbf{a}) = \left(\sum_{i=1}^n \gamma_i a_i^\rho \right)^{\kappa/\rho}.$$

We begin by establishing strictness of best responses. The second derivative of $Y(\cdot)$ with respect to a_i is

$$\frac{\partial^2 Y}{\partial a_i^2} = \gamma_i \kappa a_i^{\rho-2} \left(\sum_{\ell=1}^n \gamma_\ell a_\ell^\rho \right)^{\frac{\kappa}{\rho}-2} \left[\gamma_i (\kappa - 1) a_i^\rho + (\rho - 1) \sum_{j \neq i} \gamma_j a_j^\rho \right].$$

Thus, the second derivative of $\mathcal{U}_i$ with respect to a_i , as defined in (A.27), is strictly negative when $P(\cdot)$ is concave, $\kappa \in (0, 1]$ and $\rho \leq 1$. Under these conditions, each agent's utility is strictly concave in own action, implying that the best responses are unique.

We next establish the stability of equilibrium actions at the optimal contract. We show that when $\kappa < 2$, a unique Nash equilibrium exists and is differentiable with respect to the contract $\boldsymbol{\tau}$. The result relies on the following characterization: given $\boldsymbol{\tau}$, a positive equilibrium exists if and only if there is a solution to

$$Y^{\frac{2-\kappa}{\kappa}} \cdot \frac{1}{P'(Y)} = \kappa \left(\sum_{i=1}^n \gamma_i (\tau_i \gamma_i)^{\frac{\rho}{2-\rho}} \right)^{\frac{2-\rho}{\rho}}. \quad (\text{A.29})$$

To see why, suppose an equilibrium $\mathbf{a}^*$ exists and let Y^* denote the corresponding team

performance. Agents' actions must satisfy the equilibrium first-order conditions

$$a_i^* = \kappa \tau_i \gamma_i P'(Y^*) (Y^*)^{\frac{\kappa-\rho}{\kappa}} (a_i^*)^{\rho-1}.$$

Taking equilibrium action a_i^* to the left-hand side and taking the power of that equation to $\rho/(2-\rho)$ yields

$$(a_i^*)^\rho = (\kappa \tau_i \gamma_i P'(Y^*))^{\frac{\rho}{2-\rho}} (Y^*)^{\frac{\kappa-\rho}{\kappa} \cdot \frac{\rho}{2-\rho}} \quad \text{for every } i \in \mathcal{N}.$$

Multiplying the equation by γ_i and summing across all agents gives

$$(Y^*)^{\rho/\kappa} = (Y^*)^{\frac{\kappa-\rho}{\kappa} \cdot \frac{\rho}{2-\rho}} (\kappa P'(Y^*))^{\frac{\rho}{2-\rho}} \sum_{i=1}^n \gamma_i (\tau_i \gamma_i)^{\frac{\rho}{2-\rho}}.$$

Eq. (A.29) then follows by moving the terms involving Y^* and $P'(Y^*)$ to the left-hand side and taking the power of that equation to $(2-\rho)/\rho$. For the converse direction, suppose there exists a solution to (A.29) which we denote by Y^* . The action profile defined by

$$a_i^* = (\kappa \tau_i \gamma_i P'(Y^*))^{\frac{1}{2-\rho}} (Y^*)^{\frac{\kappa-\rho}{\kappa} \cdot \frac{1}{2-\rho}} \quad \text{for every } i \in \mathcal{N},$$

then constitutes a Nash equilibrium. This follows because when $\rho \leq 1$ and $\kappa \in (0, 1]$, each agent's utility is strictly concave in own action, so any profile satisfying the first-order conditions constitutes a Nash equilibrium.

By assumption that $\kappa < 2$, the term $Y^{(2-\kappa)/\kappa}$ is strictly increasing in Y , and since $P'(\cdot)$ is weakly decreasing by concavity of $P(\cdot)$, the left-hand side of (A.29) is strictly increasing. It follows that equation (A.29) admits a unique positive solution for any given contract τ .

Because $P(\cdot)$ is smooth, the left-hand side of (A.29) is smooth, implying that the solution $Y^*(\tau)$ is differentiable with respect to the contract τ . The statement of the result follows.

A.1.12 Proof of Proposition 4

Proof of Part (a): Recall that, after applying the monotone transformation $\tilde{Y} = \log Y$ and $\tilde{P}(\tilde{Y}) = P(\exp(\tilde{Y}))$, team performance can be written as

$$\tilde{Y}(\mathbf{a}) = \sum_{i \in \mathcal{N}} \gamma_i \log(a_i).$$

Consider an optimal contract $\boldsymbol{\tau}^*$ and equilibrium $\mathbf{a}^*$. Each agent's first-order condition is

$$a_i^* = \frac{\tau_i \gamma_i \tilde{P}'(\tilde{Y}^*)}{a_i^*}. \quad (\text{A.30})$$

We now compute the quantities in Theorem 1. Differentiating $\tilde{Y}$, we compute productivity to be $\tilde{\alpha}_i = \gamma_i/a_i^*$. Since the transformed problem is separable, the matrix $\frac{\partial^2 \tilde{Y}}{\partial a_i \partial a_j}$ of spillovers is diagonal with entries $\frac{\partial^2 \tilde{Y}}{\partial a_i^2} = -\frac{\gamma_i}{a_i^2}$. We next compute agents' centralities at equilibrium to be

$$\begin{aligned} \tilde{c}_i &= \tilde{\alpha}_i \left(1 + \frac{\tau_i \tilde{P}'(\tilde{Y}^*) \gamma_i}{(a_i^*)^2} \right)^{-1}, \\ &= \frac{\tilde{\alpha}_i}{2} \quad \text{by (A.30)}. \end{aligned}$$

We can now apply Theorem 1, which states that under any optimal contract satisfying Assumption 1, the quantity $\frac{\tilde{\alpha}_i^2}{2}$ is equal for all agents and thus all agents have the same productivity. This tells us that equilibrium actions are proportional to γ_i , each agent's factor share in the original production function. Given this, Equation (A.30) implies that each agent's payment τ_i under the optimal contract is proportional to his factor share γ_i .

Fact 4. If $\rho = 1$, only agents with maximal γ_i receive positive payments at an optimal contract.

Proof. When $\rho = 1$, transformed production is $\tilde{Y}(\mathbf{a}) = \sum_i \gamma_i a_i$, so $\tilde{\alpha}_i = \tilde{c}_i = \gamma_i$. For any paid agent i , the principal's first-order condition gives $lD\tilde{P}'(\tilde{Y}^*)\gamma_i^2 = \tilde{P}(\tilde{Y}^*) > 0$, so $lD\tilde{P}'(\tilde{Y}^*) > 0$. From this it follows that all agents who are paid must have identical γ_i , and only those with the maximal γ_i can be paid. $\square$

Proof of Part (b) for $\rho < 1$: It is again useful to transform team production: we consider an equivalent transformed problem in which we replace Y with $\tilde{Y} = \frac{1}{\rho}Y^{\rho/\kappa}$ and $P(Y)$ with $\tilde{P}(\tilde{Y}) = P((\rho\tilde{Y})^{\kappa/\rho})$. The transformed problem is:

$$\tilde{Y}(\mathbf{a}) = \frac{1}{\rho} \sum_{i=1}^n \gamma_i a_i^\rho. \quad (\text{A.31})$$

This transformation again does not change the optimal contract or the corresponding equilibrium actions. The role of ρ in the transformation is to ensure that $\tilde{Y}$ is an increasing function of actions.

We characterize how the optimal contract divides payments among agents who do receive positive payments. Each of these agents has first-order conditions

$$a_i^* = \tau_i \tilde{P}'(\tilde{Y}^*) \gamma_i \cdot (a_i^*)^{\rho-1}. \quad (\text{A.32})$$

We next compute productivities and centralities. Differentiating $\tilde{Y}$, productivities are

$$\tilde{\alpha}_i = \gamma_i a_i^{\rho-1}.$$

The matrix $\frac{\partial^2 \tilde{Y}}{\partial a_i \partial a_j}$ of spillovers is again diagonal with entries $\frac{\partial^2 \tilde{Y}}{\partial a_i^2} = \gamma_i (\rho-1) a_i^{\rho-2}$. So agents' centralities at equilibrium are

$$\begin{aligned} \tilde{c}_i &= \tilde{\alpha}_i \left(1 - \tau_i \tilde{P}'(\tilde{Y}^*) \gamma_i (\rho-1) \cdot (a_i^*)^{\rho-2} \right)^{-1} \\ &= \frac{\tilde{\alpha}_i}{2-\rho} \text{ by eq. (A.32).} \end{aligned}$$

We can now apply [Theorem 1](#), which states that under any optimal contract satisfying [Assumption 1](#), the quantity $\frac{\tilde{\alpha}_i^2}{2-\rho}$ is equal for all agents and thus all agents have the same productivity. This tells us that $(a_i^*)^{1-\rho}$ is proportional to γ_i . Given this, [\(A.32\)](#) implies that each agent's payment τ_i under the optimal contract is proportional to $\gamma_i^{\frac{1}{1-\rho}}$.

A.2 Applicability of the First-Order Approach

A key assumption in our analysis is that when we perturb the optimal contract in any direction, the induced equilibrium varies in a differentiable way ([Assumption 1](#)). We now discuss sufficient conditions for this assumption and provide balance conditions that hold in its absence.

We begin by observing that [Assumption 1](#) holds as long as the optimal contract induces a strict equilibrium and a stability property is satisfied. By the implicit function theorem, [Assumption 1](#) is implied by the following two conditions:

Assumption 6. (a) (*Invertibility of utility Hessian*) The matrix

$$\left(\frac{\partial^2 \mathcal{U}_i}{\partial a_j \partial a_i} \right)_{i,j}$$

is non-singular at contract $\boldsymbol{\tau}^*$ and corresponding equilibrium $\boldsymbol{a}^*$.

(b) (*Strictness*) The equilibrium $\boldsymbol{a}^*(\boldsymbol{\tau}^*)$ is strict.

This assumption gives more explicit conditions that guarantee a first-order approach applies. Part (a) is weaker than requiring stability of equilibrium under best-reply dynamics.³ Part (b) can impose more substantive restrictions, as we now discuss.

[Section A.2.1](#) shows we can extend the sufficient conditions for strictness from [Rogerson \[1985\]](#) to our multi-agent setting. [Section A.2.2](#) presents an analogous balance condition which only requires differentiability in *some* directions, and therefore can provide insights even when equilibrium is not strict under the optimal contract.

A.2.1 Sufficient conditions for strictness of equilibrium

This section establishes conditions on the environment that guarantee strictness of equilibrium actions. Specifically, equilibrium actions are strict when the probability functions

³If we solve for local best-reply dynamics the Jacobian of the agents' first-order conditions in others' actions is given by the Hessian above. Stability requires that this Jacobian be well-defined and have some further properties.

satisfy the *monotone likelihood ratio property* and the *convexity of distribution function property* (as in Rogerson [1985]), and when the team performance is concave in each agent's action.

We can assume without loss of generality that the possible outcomes are $S = \{1, \dots, |S|\}$ and that outcomes are ordered in increasing value to the principal: $v_1 < v_2 < \dots < v_{|S|}$. We will use two conditions from Rogerson [1985]:

Assumption 7. *The collection of distribution functions $\{P_s(\cdot)\}_{s \in S}$ satisfies:*

- (a) **Monotone Likelihood Ratio Property (MLRP):** *the ratio $P'_s(Y)/P_s(Y)$ is strictly increasing in the outcome s for any team performance.*
- (b) **Convexity of distribution function condition (CDFC):** *for any outcome s , the cumulative distribution*

$$\sum_{k=1}^s P_k(Y)$$

is convex in team performance.

MLRP requires that low-valued outcomes become less likely as team performance increases. A key implication is that the cumulative distribution function of any outcome is weakly decreasing in team performance. CDFC strengthens this by requiring convexity of the cumulative distribution, which will be crucial for the strictness of best responses.

A comparison with the single-agent setting is instructive. Rogerson [1985] shows that with one agent taking a one-dimensional action, these conditions are sufficient for a first-order approach. We are able to extend the analysis to multi-agent settings because team performance in our framework depends only on a one-dimensional aggregate of individual actions. To do so, we will require Assumption 7 as well as a concavity assumption on $Y(\mathbf{a})$. The assumption on $Y(\mathbf{a})$ is satisfied automatically in our network games setting and is satisfied for Cobb-Douglas and CES production functions whenever all factor shares are at most 1.

Theorem 10. *Suppose the collection of outcome probability functions $\{P_s(\cdot)\}_{s \in S}$ satisfies*

Assumption 7 and the conditions in Fact 1 are satisfied. Additionally, assume that the team performance function $Y(\cdot)$ is concave in each agent's action, i.e., the second derivative $\frac{\partial^2 Y}{\partial a_i^2}$ is non-positive for every agent $i \in \mathcal{N}$. Then there exists an optimal contract τ^* such that the resulting equilibrium action profile $\mathbf{a}^*(\tau^*)$ is strict.

Recall that the principal chooses a contract to maximize expected profit, taking into account the agents' equilibrium actions in response. The principal's problem, denoted by (PM), is given by:

$$\begin{aligned} \max_{\tau, \mathbf{a}^*(\tau)} \quad & \sum_{s=1}^{|S|} \left(v_s - \sum_{i \in \mathcal{N}} \tau_i(s) \right) P_s(Y), \\ \text{s.t.} \quad & \mathbf{a}^*(\tau) \text{ is an equilibrium,} \\ & \tau_i(s) \geq 0 \quad \text{for every } i, \text{ and outcome } s. \end{aligned} \tag{PM}$$

To prove Theorem 10, we begin by analyzing a relaxation of the principal's profit maximization problem. In this relaxed formulation, the principal chooses a contract τ and an action profile $\mathbf{a}^*(\tau)$ that satisfies the first-order (stationary) conditions of the agents' utility maximization problems. The objective is to maximize expected profits subject to these first-order conditions. The relaxation, denoted by (RPM), is given by:

$$\begin{aligned} \max_{\tau, \mathbf{a}^*(\tau)} \quad & \sum_{s=1}^{|S|} \left(v_s - \sum_{i \in \mathcal{N}} \tau_i(s) \right) P_s(Y), \\ \text{s.t.} \quad & \frac{\partial \mathcal{U}_i}{\partial a_i} = 0 \quad \text{for every } i \in \mathcal{N}, \\ & \tau_i(s) \geq 0 \quad \text{for every } i, \text{ and outcome } s. \end{aligned} \tag{RPM}$$

We now justify the use of this relaxation. Any optimal solution to (PM) is a feasible solution to (RPM). To see why, any optimal action for each agent must satisfy that agent's first-order condition. Additionally, any agent that takes a zero action must be paid zero at the optimal contract (else the principal could induce the same equilibrium by providing zero payments to these agents, which would strictly increase profit). By our assumption that the marginal cost at a zero action is zero, such an agent's marginal utility is zero. Thus, all equilibrium

actions—whether interior or at the boundary—satisfy the first-order conditions that define the constraint in (RPM).

The primary challenge lies in establishing that an optimal solution to (RPM) is a feasible solution to (PM). Consequently, by the justification in the paragraph above, it would be an optimal solution to the latter. Toward this end, we will show that each agent’s utility function is strictly concave at any optimal solution to (RPM). Feasibility in (PM) and strictness of the equilibrium action profile will follow.

We begin with a lemma showing it is optimal to pay agents more at better outcomes:

Lemma 8. *Consider any optimal solution $(\tau^*, \mathbf{a}^*(\tau^*))$ to (RPM). For any agent $i \in \mathcal{N}$,*

$$\tau_i^*(s+1) \geq \tau_i^*(s) \quad \text{for all } s \in \mathcal{S}.$$

Furthermore, for any agent that receives a positive payment, there must be a strict increase in payments at some outcome.

Proof. Consider any outcome $s \in \mathcal{S}$ that provides a payment to *some* agent i . The principal’s KKT first-order condition implies

$$\frac{1}{u'_i(\tau_i^*(s))} = \lambda_i \frac{\partial Y}{\partial a_i} \frac{P'_s(Y)}{P_s(Y)}, \quad (\text{A.33})$$

where λ_i is the Lagrange multiplier corresponding to that agent’s marginal utility constraint. We will establish that $\lambda_i > 0$ for any agent that receives a payment. Suppose this was not the case. Then, any outcome at which this agent is paid, it must be $P'_s(Y) < 0$ (the left hand side of (A.33) is always positive). But this means that the agent must be taking a zero action because the marginal utility is non-positive. This contradicts optimality because the principal can induce the same equilibrium action by giving this agent zero payments and obtain a strictly higher profit. Thus, $\lambda_i > 0$ for any agent that receives a payment.

For the rest of the argument, it suffices to focus on these agents — payments are trivially monotonic for an agent that does not receive a payment at *any* outcome.

Consider any agent i and suppose $\tau_i^*(s) > 0$, but $\tau_i^*(s+1) = 0$. We will show that the principal can strictly improve profits by providing a payment to agent i at outcome $(s+1)$. Recall, the KKT first-order condition for $\tau_i^*(s)$, established in (A.33), is

$$1 - u'_i(\tau_i^*(s))\lambda_i \frac{\partial Y}{\partial a_i} \frac{P'_s(Y)}{P_s(Y)} = 0$$

By assumption, $\tau_i^*(s+1) < \tau_i^*(s)$ and the outcomes satisfy MLRP, that is,

$$\frac{P'_{s+1}(Y)}{P_{s+1}(Y)} > \frac{P'_s(Y)}{P_s(Y)}.$$

Combined with the observation $\lambda_i > 0$, we get

$$1 - u'_i(\tau_i^*(s+1))\lambda_i \frac{\partial Y}{\partial a_i} \frac{P'_{s+1}(Y)}{P_{s+1}(Y)} < 0. \quad (\text{A.34})$$

The derivative of the Lagrangian w.r.t $\tau_i(s+1)$ is

$$\frac{\partial \mathcal{L}}{\partial \tau_i(s+1)} = -P_{s+1}(Y) \left(1 - u'_i(\tau_i(s+1))\lambda_i \frac{\partial Y}{\partial a_i} \frac{P'_{s+1}(Y)}{P_{s+1}(Y)} \right) > 0,$$

where the inequality follows from (A.34). This contradicts the optimality of $\tau_i^*(s+1) = 0$, so $\tau_i^*(s+1) > 0$. Moreover, under MLRP and the concavity of $u_i(\cdot)$, analysis of (A.33) implies that $\tau_i^*(s+1) \geq \tau_i^*(s)$. $\square$

The intuition behind the result is straightforward. Agents are more responsive to incentives at outcomes that become more likely as team performance improves. Since the principal's value also increases with the outcome, this alignment leads the principal to offer stronger incentives at higher-valued outcomes.

We utilize the monotonicity result to prove [Theorem 10](#). A key step establishes that each agent's utility function is strictly concave at any optimal solution to (RPM).

Proof of Theorem 10. Consider any optimal solution $(\boldsymbol{\tau}^*, \mathbf{a}^*(\boldsymbol{\tau}^*))$ to (RPM). An argument identical to the proof of [Fact 1](#) establishes that an optimal solution to (RPM) exists under

the conditions of [Fact 1](#). We can write the utility of an agent as

$$\begin{aligned}\mathcal{U}_i &= \sum_{s=1}^{|S|} u_i(\tau_i(s)) P_s(Y) - C_i(a_i) \\ &= u_i(\tau_i(1)) + \sum_{s=2}^{|S|} \underbrace{(u_i(\tau_i(s)) - u_i(\tau_i(s-1)))}_{u_i^\Delta(\tau_i(s))} \sum_{k=s}^{|S|} P_k(Y) - C_i(a_i).\end{aligned}$$

The second derivative of $\mathcal{U}_i$ with respect to a_i is

$$\frac{\partial^2 \mathcal{U}_i}{\partial a_i^2} = \frac{\partial^2 Y}{\partial a_i^2} \sum_{s=2}^{|S|} u_i^\Delta(\tau_i(s)) \sum_{k=s}^{|S|} P'_k(Y) + \left(\frac{\partial Y}{\partial a_i} \right)^2 \sum_{s=2}^{|S|} u_i^\Delta(\tau_i(s)) \sum_{k=s}^{|S|} P''_k(Y) - C''_i(a_i).$$

We will argue that the second-derivative is strictly negative for any action profile. By [Lemma 8](#), each $u_i^\Delta(\tau_i(s))$ is non-negative. By concavity of $Y(\cdot)$ in each agent's action and MLRP, the first term is non-positive. By CDFC, the second term is non-positive. And finally, since the cost function is strictly convex, $C''_i(a_i) > 0$. This establishes the fact that the second derivative is strictly negative. Thus, the utility function of any agent is strictly concave. This has two important implications. First, any action profile where all agents' first-order conditions are satisfied is in fact an equilibrium and thus a feasible solution to [\(PM\)](#). So any optimal solution to [\(RPM\)](#) is an optimal contract. Second, the equilibrium action profile at this optimal contract is strict. The statement of the result follows. $\square$

A.2.2 Directional differentiability

This section shows modified balance conditions can continue to hold when the induced equilibrium varies differentially as the contract is perturbed in some, but not necessarily all, directions. The underlying idea is that the principal may only have available perturbations that preserve some global constraints along with the first-order conditions.

Consider an optimal contract $\boldsymbol{\tau}^*$ and a corresponding equilibrium $\mathbf{a}^*(\boldsymbol{\tau}^*)$ that is stable but not strict. Then the implicit function theorem ensures that as we perturb the contract $\boldsymbol{\tau}^*$ to $\boldsymbol{\tau}$ locally, there is an action profile $\mathbf{a}(\boldsymbol{\tau})$ satisfying agents' first-order conditions that is continuously differentiable in $\boldsymbol{\tau}$. Since the equilibrium is no longer strict, however, these

action profiles need not be equilibria for all τ : if agent i is indifferent to his equilibrium action $a^*(\tau^*)_i$ and some alternative, then after perturbing the contract the locally optimal action $a(\tau)_i$ need no longer be globally optimal. Then there could be perturbations for which the principal would prefer $\mathbf{a}(\tau)$ to $\mathbf{a}^*(\tau^*)$, so our full balance conditions need no longer hold. But if there are perturbations such that $\mathbf{a}(\tau)$ remains an equilibrium, then we can obtain balance conditions in the directions of all such perturbations.

Consider increasing payments to agent i in the direction $\mathbf{t}_i$, where $t_i(s)$ specifies the change in payments to agent i in each outcome s . We hold fixed payments to all other agents $j \neq i$.

Definition. We say **equilibrium is maintained** in direction $\mathbf{t}_i$ if there exist a one-parameter family $\tau(x)$ of contracts with $\tau(0) = \tau^*$ such that $\tau'_i(0) = \mathbf{t}_i$, $\tau'_j(0) = 0$ for $j \neq i$, and such that $\mathbf{a}(\tau(x))$ is an equilibrium for x in some neighborhood of 0.

When there are directions where equilibrium is maintained, we obtain balance conditions in those directions. The statements of these conditions must be modified since we now change payments under multiple outcomes.

Theorem 11. Suppose τ^* is an optimal contract inducing equilibrium $\mathbf{a}^*$ with team performance Y^* . For all directions $\mathbf{t}_i$ such that equilibrium is maintained and $\tau_i^*(s) > 0$ whenever $t_i(s) > 0$, we have

$$q\alpha_i c_i \left(\sum_{s \in \mathcal{S}} P'_s(Y^*) t_i(s) \cdot \frac{u'_i(\tau_i^*(s))}{C'''_i(a_i^*)} \right) = \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s)$$

for a constant q that is independent of the direction $\mathbf{t}_i$ and agent i .

Proof. The proof follows a similar approach to the proof of [Theorem 1](#). Consider any agent i receiving a payment in at least one outcome. Also, consider a direction of payment perturbation $\mathbf{t}_i$. As before, we can ignore agents that receive a zero payment at the optimal contract as they continue to take a zero action and not contribute to the change in equilibrium team

performance. The change induced by such perturbation $\mathbf{t}_i$ is

$$dY(\mathbf{t}_i) = \nabla Y(\mathbf{a}^*)^\top \cdot d\mathbf{a}^*(\mathbf{t}_i),$$

where $dY(\cdot)$ is the directional derivative of equilibrium team performance and $d\mathbf{a}^*(\cdot)$ is the directional derivative of equilibrium actions. We state a lemma which characterizes the change in equilibrium actions for such a perturbation.

Lemma 9. *The change in equilibrium team performance as payments to agent i are perturbed in direction $\mathbf{t}_i$ is*

$$dY(\mathbf{t}_i) = l\alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y^*) t_i(s) \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)},$$

for some constant l .

Proof. Consider the equilibrium action profile $\mathbf{a}^*$. For an agent j , the first-order conditions imply a_j^* must solve the equation

$$C'_j(a_j) = \left(\sum_{s' \in \mathcal{S}} P'_{s'}(Y) u_j(\tau_j(s')) \right) \frac{\partial Y}{\partial a_j}. \quad (\text{A.35})$$

Let us take the directional derivative of (A.35) in direction $\mathbf{t}_i$. The expression we obtain depends on whether $j = i$. For all $j \neq i$,

$$\begin{aligned} C''_j(a_j) da_j^*(\mathbf{t}_i) &= \left(\sum_{s \in \mathcal{S}} P'_s(Y) u_j(\tau_j(s)) \right) \sum_{k=1}^n \frac{\partial^2 Y}{\partial a_k \partial a_j} da_k^*(\mathbf{t}_i) \\ &\quad + \frac{\partial Y}{\partial a_j} dY(\mathbf{t}_i) \sum_{s \in \mathcal{S}} P''_s(Y) u_j(\tau_j(s)). \end{aligned}$$

On the other hand, for $j = i$

$$\begin{aligned} C_j''(a_j) da_j^*(\mathbf{t}_i) &= \left(\sum_{s \in \mathcal{S}} P_s'(Y) u_j(\tau_j(s)) \right) \sum_{k=1}^n \frac{\partial^2 Y}{\partial a_k \partial a_j} da_k^*(\mathbf{t}_i) \\ &\quad + \frac{\partial Y}{\partial a_j} \sum_{s \in \mathcal{S}} P_s'(Y) t_j(s) u_j'(\tau_j(s)) + \frac{\partial Y}{\partial a_j} dY(\mathbf{t}_i) \sum_{s \in \mathcal{S}} P_s''(Y) u_j(\tau_j(s)). \end{aligned}$$

We can combine these equations to write the resulting expression in vector form as

$$d\mathbf{a}^*(\mathbf{t}_i) = [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \begin{bmatrix} 0 \\ \frac{\partial Y}{\partial a_i} \cdot \sum_{s \in \mathcal{S}} P_s'(Y) t_i(s) u_i'(\tau_i(s)) \\ 0 \end{bmatrix} + dY(\mathbf{t}_i) [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d},$$

where $\mathbf{d}$ is a vector with j^{th} element defined as

$$d_j := \frac{\partial Y}{\partial a_j} \cdot \sum_{s \in \mathcal{S}} P_s''(Y) u_j(\tau_j(s)).$$

The change in equilibrium team performance is

$$\begin{aligned} dY(\mathbf{t}_i) &= \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \begin{bmatrix} 0 \\ \frac{\partial Y}{\partial a_i} \cdot \sum_{s \in \mathcal{S}} P_s'(Y) t_i(s) u_i'(\tau_i(s)) \\ 0 \end{bmatrix} \\ &\quad + dY(\mathbf{t}_i) \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}. \end{aligned}$$

Applying the definitions of α_i and c_i while rearranging the equation, we obtain

$$\begin{aligned} dY(\mathbf{t}_i) &= \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}} \alpha_i c_i \sum_{s \in \mathcal{S}} P_s'(Y^*) t_i(s) \cdot \frac{u_i'(\tau_i(s))}{C_i''(a_i^*)}, \\ &= l \alpha_i c_i \sum_{s \in \mathcal{S}} P_s'(Y^*) t_i(s) \cdot \frac{u_i'(\tau_i(s))}{C_i''(a_i^*)}, \end{aligned}$$

where $l = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}}$. Observing that l does not depend on i , we obtain the desired result. $\square$

To prove the theorem, we utilize this expression for change in team performance to analyze the principal's first-order condition. For any agent i , consider a direction of perturbation $\mathbf{t}_i$ such that whenever $t_i(s) > 0$, the optimal contract is such that $\tau_i^*(s) > 0$. Because equilibrium is maintained in direction $\mathbf{t}_i$, the following principal first-order condition must hold:

$$dY(\mathbf{t}_i) \underbrace{\sum_{s \in \mathcal{S}} \left(v_s - \sum_{i=1}^n \tau_i(s) \right) P'_s(Y^*)}_D = \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s)$$

Substituting the expression in [Lemma 9](#), we get

$$lD\alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y^*) t_i(s) \cdot \frac{u'_i(\tau_i^*(s))}{C'''_i(a_i^*)} = \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s).$$

The expression in the theorem follows by taking $q = lD$. □

A.3 Contract design under measurement error

Our analysis has assumed the principal knows the production function $Y(\mathbf{a})$. In this section we ask the following question: when can the principal improve on a given contract using only an imperfect measurement of a team's network structure? More concretely: the principal can check our balance condition using the observed technological complementarity matrix. Suppose that this suggests shifting incentives from one agent j to another agent i would be beneficial. Will this shift actually increase the principal's payoff under the true network?

We continue to work in the setting of [Section 1.2.1](#) with linear-quadratic production and risk-neutral agents:

$$Y(\mathbf{a}) = \sum_{i=1}^n k_i a_i + \frac{1}{2} \sum_{i,j=1}^n G_{ij} a_i a_j.$$

Suppose a principal observes productivities $\boldsymbol{\alpha}$, the derivative $P'(Y^*)$, and an imperfect measurement $\mathbf{G}'$ of $\mathbf{G}$. Furthermore, suppose the principal naively calculates the centrality

vector using the measured technological complementarity matrix $\mathbf{G}'$ as if it were $\mathbf{G}$:

$$\mathbf{c}' = \boldsymbol{\alpha}^\top (I - P'(Y^*)\mathbf{T}\mathbf{G}')^{-1}.$$

Suppose that under this calculation, the balance condition suggests increasing τ_i and decreasing τ_j will be profitable, i.e.,

$$\alpha_i c'_i - \alpha_j c'_j > \epsilon \quad (\text{A.36})$$

for some $\epsilon > 0$. We ask when the principal can guarantee that the analogous statement holds for the quantities derived from the true network, i.e.,

$$\alpha_i c_i - \alpha_j c_j > 0.$$

This inequality implies that marginally increasing τ_i and decreasing τ_j will indeed increase revenue.

The measurement of the network clearly needs to have enough precision, i.e., the difference between $\mathbf{G}$ and $\mathbf{G}'$ should be small in some sense. Our (standard) measure of this difference will be the spectral norm $\|\mathbf{G} - \mathbf{G}'\|_2$. Our question then becomes: is the principal safe in taking the approximation at face value if this difference is small? Though the answer turns out to be no in general, we can give several sufficient conditions. The first is that the spillovers between agents are not very large, as we now discuss.

Bounded spillovers: Suppose that $\|\mathbf{T}\mathbf{G}\|_2 P'(Y^*) < r$ for some fixed constant $r < 1$. The assumption says that spillovers between agents are not too large: there is a bound on the total effects of direct and indirect spillovers. Under this condition, the next result shows that the principal can ignore small amounts of measurement error in checking for profitable deviations. Moreover, the proof gives an explicit bound on how much measurement error is permissible.

Proposition 16. *Suppose $\|\mathbf{T}\mathbf{G}\|_2 P'(Y^*) < r$ for some fixed constant $r < 1$. If $\alpha_i c'_i - \alpha_j c'_j > \epsilon$ for some $\epsilon > 0$, then we can choose $\delta > 0$ independent of $\mathbf{G}$ such that $\alpha_i c_i - \alpha_j c_j > 0$ whenever $\|\mathbf{G} - \mathbf{G}'\|_2 < \delta$.*

Proof. We take δ small enough so that $r + \delta P'(0) < 1$, and therefore $r + \delta P'(Y) < 1$ for all Y by concavity of $P(Y)$. Note that

$$\underbrace{(I - P'(Y^*)\mathbf{T}\mathbf{G})^{-1}}_{\mathbf{M}} - \underbrace{(I - P'(Y^*)\mathbf{T}\mathbf{G}')^{-1}}_{\mathbf{M}'} = (I - P'(Y^*)\mathbf{T}\mathbf{G})^{-1}(P'(Y^*)\mathbf{T}(\mathbf{G} - \mathbf{G}'))(I - P'(Y^*)\mathbf{T}\mathbf{G}')^{-1}.$$

Solving for the norm of $\mathbf{M} - \mathbf{M}'$,

$$\begin{aligned} \|\mathbf{M} - \mathbf{M}'\|_2 &= \|(I - P'(Y^*)\mathbf{T}\mathbf{G})^{-1}P'(Y^*)\mathbf{T}(\mathbf{G} - \mathbf{G}')(I - P'(Y^*)\mathbf{T}\mathbf{G}')^{-1}\|_2, \\ &\leq \frac{\delta P'(Y^*)}{(1-r)(1-r-\delta P'(Y^*))} \quad (\text{by } \max_i \tau_i \leq 1), \\ &\leq \frac{\delta P'(0)}{(1-r)(1-r-\delta P'(0))} \quad (\text{by concavity of } P(Y)). \end{aligned}$$

This implies

$$\|\mathbf{c} - \mathbf{c}'\|_2 \leq \|\boldsymbol{\alpha}\|_2 \cdot \frac{\delta P'(0)}{(1-r)(1-r-\delta P'(0))}.$$

So

$$\|\boldsymbol{\alpha}(\mathbf{c} - \mathbf{c}')\|_2 \leq \|\boldsymbol{\alpha}\|_2^2 \cdot \frac{\delta P'(0)}{(1-r)(1-r-\delta P'(0))}.$$

The result follows from choosing δ small enough so that the right-hand side is less than ϵ . $\square$

When spillovers are very large, however, productivities and centralities can be very sensitive to small perturbations of the network. In this case, a violation of the balance condition based on the observed network need not give a profitable deviation for the principal. The following example shows that when spillovers are very large, the effects of transferring bonuses between agents may be very sensitive to mismeasurement.

Example. Consider a network with four agents and two non-zero links G_{12} and G_{34} , with

$$Y = \sum_i a_i + G_{12}a_1a_2 + G_{34}a_3a_4.$$

Suppose that agents are risk-neutral and $P(Y)$ is locally linear with slope $P'(Y^*)$. By taking Y^* close to zero, we can take $P'(Y^*)$ to be arbitrarily large.

The network consists of two distinct subteams with a single link in each. We now show that small measurement error can generate large observed differences in centralities between the two subteams.

Consider a contract $\tau = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. Equilibrium actions are

$$a_1^* = a_2^* = \frac{P'(Y^*)}{8} \cdot \frac{1}{1 - P'(Y^*)G_{12}/8}, a_3^* = a_4^* = \frac{P'(Y^*)}{8} \cdot \frac{1}{1 - P'(Y^*)G_{34}/8}.$$

The principal observes $\mathbf{G}'$ and the productivities

$$\alpha_1 = \alpha_2 = 1 + G_{12}a_1^*, \alpha_3 = \alpha_4 = 1 + G_{34}a_3^*.$$

We suppose that $\mathbf{G}'$ also has two non-zero links G'_{12} and G'_{34} (and no other links). So the principal naively calculates centralities

$$\mathbf{c}'_1 = \mathbf{c}'_2 = \alpha_1 \frac{1}{1 - P'(Y^*)G'_{12}/8}, \mathbf{c}'_3 = \mathbf{c}'_4 = \alpha_3 \frac{1}{1 - P'(Y^*)G'_{34}/8}.$$

Fix $\epsilon > 0$. Let

$$G_{12} = g + \eta/2, G_{34} = g, G'_{12} = g, G'_{34} = g + \eta,$$

where $\eta > 0$ is arbitrary. So the link between 1 and 2 is stronger than the link between 3 and 4, but the principal believes the link between 3 and 4 is the stronger one due to measurement error. We calculate

$$\begin{aligned} \alpha_1 \mathbf{c}'_1 &= \left(1 + \frac{P'(Y^*)}{8} \cdot \frac{g + \eta/2}{1 - P'(Y^*)(g + \eta/2)/8} \right)^2 \cdot \frac{1}{1 - P'(Y^*)g/8} \\ &< \left(1 + \frac{P'(Y^*)}{8} \cdot \frac{g}{1 - P'(Y^*)g/8} \right)^2 \cdot \frac{1}{1 - P'(Y^*)(g + \eta)/8} \\ &= \alpha_3 \mathbf{c}'_3 \end{aligned}$$

for g such that $P'(Y^*)g/8$ is sufficiently close to 1. By taking g such that $P'(Y^*)g/8$ is sufficiently close to 1, we can ensure $\alpha_3 \mathbf{c}'_3 - \alpha_1 \mathbf{c}'_1 \geq \epsilon$. Since $\alpha_1 > \alpha_3$ and $c_1 > c_3$, however, $\alpha_3 c_3 - \alpha_1 c_1 < 0$. This all holds taking η , and therefore the norm $\|\mathbf{G} - \mathbf{G}'\|_2$, to be arbitrarily small.

The key issue in the example is that the principal must decide how to allocate incentives between two disconnected sub-teams. Spillovers in each of these teams are strong, so a small error in measuring the network within a team can be amplified substantially. This makes identifying which subteam is more responsive to incentives important, but measurement error may make it impossible. We can rule out this situation with a standard assumption ensuring the network cannot be partitioned into two subteams with only very weak connections between them.

Bounded Spectral Gap: Suppose that the spectral gap (i.e., the difference between the absolute values of the largest and second-largest eigenvalues) of $\mathbf{T}\mathbf{G}$ is bounded below by some positive constant. (This is a standard measure of segregation—see, e.g., [von Luxburg \[2007\]](#) and [Golub and Jackson \[2012\]](#).) Then the principal can adapt the balance condition to ensure violations guarantee profitable changes in contract as long as the measurement error $\|\mathbf{G} - \mathbf{G}'\|_2$ is not too large.

The key idea is that the principal decomposes $\mathbf{G}'$ into a leading principal component and an orthogonal residual component:

$$\mathbf{G}' = \mathbf{L}' + \mathbf{R}'.$$

The apparent value of reallocating incentives from agent i to agent j , as in (A.36), can now be calculated as the sum of two parts: (i) the contribution to this based on the leading component of the network and (ii) the contribution from the residual part of the network, which captures the group structure of the network. The argument shows that (i) can be very sensitive to mismeasurement, but that mismeasurement will only affect its magnitude and not its sign. The contribution of (ii) is not very sensitive to mismeasurement because the spectral gap is not too small, which ensures the spillovers in the residual components are bounded.

To formalize this, let $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\lambda_1, \dots, \lambda_n$ be the eigenvectors and eigenvalues, respectively, of $\mathbf{T}^{\frac{1}{2}}\mathbf{G}\mathbf{T}^{\frac{1}{2}}$, chosen so that the eigenvalues satisfy $|\lambda_1| > \dots > |\lambda_n|$, the eigenvectors form an orthonormal basis, and $\mathbf{v}_1$ is normalized to have positive entries. Let $\mathbf{v}'_1, \dots, \mathbf{v}'_n$ and $\lambda'_1, \dots, \lambda'_n$ be the eigenvectors and eigenvalues, respectively, of $\mathbf{T}^{\frac{1}{2}}\mathbf{G}'\mathbf{T}^{\frac{1}{2}}$, chosen in the same way. The final result of this section shows that when measurement error is small, the principal can confirm a deviation be profitable by checking versions of the balance condition

in the leading principal component $\mathbf{L}'$ and the residual component $\mathbf{R}'$. We note that the orthogonal transformation to the eigenbasis introduces some additional terms related to the contract $\boldsymbol{\tau}$.

Proposition 17. *Suppose $P'(Y^*)\lambda_1 - P'(Y^*)|\lambda_2| > r'$ for some constant $r' > 0$. If*

$$\tau_i^{-\frac{1}{2}}\alpha_i v'_{1i} - \tau_j^{-\frac{1}{2}}\alpha_j v'_{1j} > \epsilon \text{ and } \sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}} \mathbf{c}', \mathbf{v}'_{\ell} \rangle (\tau_i^{-\frac{1}{2}}\alpha_i v'_{\ell i} - \tau_j^{-\frac{1}{2}}\alpha_j v'_{\ell j}) > \epsilon$$

for some $\epsilon > 0$, then we can choose $\delta > 0$ independent of $\mathbf{G}$ such that $\alpha_i c_i - \alpha_j c_j > 0$ whenever $\|\mathbf{G} - \mathbf{G}'\|_2 < \delta$.

Proof. We will show that we can choose $\delta > 0$ independent of $\mathbf{G}$ such that

$$\tau_i^{-\frac{1}{2}}\alpha_i v_{1i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{1j} > 0 \text{ and } \sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}} \mathbf{c}, \mathbf{v}_{\ell} \rangle (\tau_i^{-\frac{1}{2}}\alpha_i v_{\ell i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{\ell j}) \geq 0.$$

This will imply

$$\begin{aligned} \alpha_i c_i - \alpha_j c_j &= \sum_{\ell=1}^n \langle \mathbf{T}^{\frac{1}{2}} \mathbf{c}, \mathbf{v}_{\ell} \rangle (\tau_i^{-\frac{1}{2}}\alpha_i v_{\ell i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{\ell j}) \\ &= \langle \mathbf{T}^{\frac{1}{2}} \mathbf{c}, \mathbf{v}_1 \rangle (\tau_i^{-\frac{1}{2}}\alpha_i v_{1i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{1j}) + \sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}} \mathbf{c}, \mathbf{v}_{\ell} \rangle (\tau_i^{-\frac{1}{2}}\alpha_i v_{\ell i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{\ell j}) \\ &> 0 \end{aligned}$$

as desired.

Step (i): $\tau_i^{-\frac{1}{2}}\alpha_i v_{1i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{1j} > 0$.

We begin by bounding $\|\mathbf{v}_1 - \mathbf{v}'_1\|_2$. We have

$$\begin{aligned} \|\mathbf{T}^{\frac{1}{2}} \mathbf{G} \mathbf{T}^{\frac{1}{2}} \mathbf{v}'_1\|_2 &\geq \|\mathbf{T}^{\frac{1}{2}} \mathbf{G}' \mathbf{T}^{\frac{1}{2}} \mathbf{v}'_1\|_2 - \|\mathbf{T}^{\frac{1}{2}} (\mathbf{G}' - \mathbf{G}) \mathbf{T}^{\frac{1}{2}} \mathbf{v}'_1\|_2, \\ &\geq \|\lambda'_1 \mathbf{v}'_1\|_2 - \|\mathbf{T}^{\frac{1}{2}}\|_2 \|\mathbf{G}' - \mathbf{G}\|_2 \|\mathbf{T}^{\frac{1}{2}}\|_2 \|\mathbf{v}'_1\|_2, \\ &\geq \lambda'_1 - \delta, \\ &\geq \lambda_1 - 2\delta. \end{aligned}$$

The final inequality follows from the bound $\|\mathbf{T}^{\frac{1}{2}}(\mathbf{G}' - \mathbf{G})\mathbf{T}^{\frac{1}{2}}\|_2 < \delta$ which implies $|\lambda_1 - \lambda'_1| < \delta$ by Weyl's theorem.

On the other hand, writing $\mathbf{v}'_1 = \sum w_i v_i$ with $\sum_i w_i^2 = 1$, we have

$$\|\mathbf{T}^{\frac{1}{2}}\mathbf{G}\mathbf{T}^{\frac{1}{2}}\mathbf{v}'_1\|_2 = \left(\sum_i w_i^2 |\lambda_i|^2 \right)^{\frac{1}{2}}.$$

Combining our two expressions for $\|\mathbf{T}^{\frac{1}{2}}\mathbf{G}\mathbf{T}^{\frac{1}{2}}\mathbf{v}'_1\|_2$,

$$\left(\sum_i w_i^2 |\lambda_i|^2 \right)^{\frac{1}{2}} \geq \lambda_1 - 2\delta.$$

Since $\lambda_1 - |\lambda_2| > r'/P'(Y^*)$ and $\lambda_1 P'(Y^*) < 1$, this implies $w_1 \rightarrow 1$ as $\delta \rightarrow 0$. So we can choose $f(\delta)$ depending only on r' and $P'(Y^*)$ and converging to 0 as $\delta \rightarrow 0$ such that $\|\mathbf{v}_1 - \mathbf{v}'_1\|_2 < f(\delta)$ whenever $\|\mathbf{G}' - \mathbf{G}\|_2 < \delta$.

Since $\tau_i^{-\frac{1}{2}}\alpha_i v'_{1i} - \tau_j^{-\frac{1}{2}}\alpha_j v'_{1j} > \epsilon$, we can choose δ sufficiently small so that

$$\tau_i^{-\frac{1}{2}}\alpha_i v_{1i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{1j} > 0$$

whenever $\|\mathbf{v}_1 - \mathbf{v}'_1\|_2 < f(\delta)$ and therefore whenever $\|\mathbf{G}' - \mathbf{G}\|_2 < \delta$.

Step (ii): $\sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}}\mathbf{c}, \mathbf{v}_\ell \rangle (\tau_i^{-\frac{1}{2}}\alpha_i v_{\ell i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{\ell j}) \geq 0$.

We can write $\mathbf{T}^{\frac{1}{2}}\mathbf{G}\mathbf{T}^{\frac{1}{2}} = \mathbf{H}_1 + \mathbf{H}_2$, where $\mathbf{H}_1$ is rank one with eigenvalue λ_1 and eigenvector $\mathbf{v}_1$. Similarly we can write $\mathbf{T}^{\frac{1}{2}}\mathbf{G}'\mathbf{T}^{\frac{1}{2}} = \mathbf{H}'_1 + \mathbf{H}'_2$. Then

$$\sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}}\mathbf{c}, \mathbf{v}_\ell \rangle \mathbf{v}_\ell = (\mathbf{I} - P'(Y^*)\mathbf{H}_2)^{-1}\mathbf{T}^{\frac{1}{2}}\boldsymbol{\alpha} - \langle \mathbf{T}^{\frac{1}{2}}\boldsymbol{\alpha}, \mathbf{v}_1 \rangle \mathbf{v}_1$$

and

$$\sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}}\mathbf{c}', \mathbf{v}'_\ell \rangle \mathbf{v}'_\ell = (\mathbf{I} - P'(Y^*)\mathbf{H}'_2)^{-1}\mathbf{T}^{\frac{1}{2}}\boldsymbol{\alpha} - \langle \mathbf{T}^{\frac{1}{2}}\boldsymbol{\alpha}, \mathbf{v}'_1 \rangle \mathbf{v}'_1.$$

We have $\|P'(Y^*)\mathbf{H}_2\|_2 \leq P'(Y^*)|\lambda_2|$. Because $P'(Y^*)\lambda_1 < 1$ and $\lambda_1 - |\lambda_2| > r' > 0$,

we can choose $r < 1$ independent of $\mathbf{G}$ such that $P'(Y^*)|\lambda_2| < r$. Since $\|\mathbf{H}_2 - \mathbf{H}'_2\|_2 \leq \|\mathbf{G} - \mathbf{G}'\|_2 + \|\mathbf{H}_1 - \mathbf{H}'_1\|_2$, essentially the same argument as in the proof of [Proposition 16](#) gives a bound on

$$\|(\mathbf{I} - P'(Y^*)\mathbf{H}_2)^{-1}\mathbf{T}^{\frac{1}{2}}\boldsymbol{\alpha} - (\mathbf{I} - P'(Y^*)\mathbf{H}'_2)^{-1}\mathbf{T}^{\frac{1}{2}}\boldsymbol{\alpha}\|_2$$

in terms of $\delta > 0$. We note the bound now includes an additional factor $\max_i \tau_i^{-\frac{1}{2}}$, which we can assume is well-defined by dropping any agents receiving zero payment.

The bound on $\|\mathbf{v}_1 - \mathbf{v}'_1\|_2$ from Step (i) implies that we can bound $\|\langle \mathbf{T}^{\frac{1}{2}}\boldsymbol{\alpha}, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{T}^{\frac{1}{2}}\boldsymbol{\alpha}, \mathbf{v}'_1 \rangle \mathbf{v}'_1\|_2$ in terms of δ .

Combining these bounds, we can bound

$$\left\| \sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}}\mathbf{c}, \mathbf{v}_\ell \rangle \mathbf{v}_\ell - \sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}}\mathbf{c}', \mathbf{v}'_\ell \rangle \mathbf{v}'_\ell \right\|_2$$

in terms of δ . So when $\delta > 0$ is sufficiently small we must have

$$\sum_{\ell > 1} \langle \mathbf{T}^{\frac{1}{2}}\mathbf{c}, \mathbf{v}_\ell \rangle (\tau_i^{-\frac{1}{2}}\alpha_i v_{\ell i} - \tau_j^{-\frac{1}{2}}\alpha_j v_{\ell j}) \geq 0$$

as desired. □

A.4 Sufficient conditions for positive payments

The balance result in [Theorem 1](#) only applies to agents receiving a positive payment under a given outcome. As discussed in [Section 1.4.2](#), not all agents necessarily receive positive payments at the optimal contract. In this section, we provide sufficient conditions on the environment which guarantee every agent receives a positive payment at some outcome.

Assumption 8. *The environment is such that:*

- (a) *The contract giving payments $\tau_i(s) = 0$ for all i and s is not optimal. Moreover, under the optimal contract $\boldsymbol{\tau}^*$, there is an agent i satisfying $\tau_i^*(s) > 0$ for some s and*

centrality $c_i > 0$.

(b) For every agent i , $\lim_{\tau \rightarrow 0} u'_i(\tau) = \infty$.

(c) For all i and j and any action profile $\mathbf{a}$,

$$\frac{\partial^2 Y(\mathbf{a})}{\partial a_j \partial a_i} \geq 0.$$

Part (a) ensures that the principal finds it optimal to pay at least one agent in the team and that there is an agent who receives a payment at some outcome and has a positive centrality. Part (b) is a standard Inada condition for the agent's utility. Part (c) says that agents' actions are complements, and performance is convex in each agent's own effort.

Under these assumptions, all agents are paid precisely at the outcomes that are more likely when team performance increases slightly.

Proposition 18. *Suppose τ^* is an optimal contract with induced team performance Y^* . For any agent i and any outcome s ,*

$$\tau_i^*(s) > 0 \quad \text{if and only if} \quad P'_s(Y^*) > 0.$$

In general, it can be optimal to exclude some agents from the optimal team (by offering them no incentives). The proposition states that when agents are sufficiently risk averse and actions are complementary, it is optimal to include all agents. Moreover, all agents are paid under all outcomes that would become more likely if they increased their effort.

The remainder of this section proves the proposition. We begin by stating a key lemma, which will help show that at any optimal contract all agents have positive centralities.

Lemma 10. *At any optimal contract τ^* , the spillover matrix $\mathbf{H}^{-1}\mathbf{U}\mathbf{G}$ has spectral radius strictly smaller than 1.*

Proof. We will use ρ to denote the spectral radius of $\mathbf{H}^{-1}\mathbf{U}\mathbf{G}$. We will show that the

spillover matrix cannot have $\rho \geq 1$ at any optimal contract. To do so, we will construct a perturbation of the contract giving the principal a higher payoff.

By definition (see [Section 1.3.1](#)),

$$\mathbf{c}^T [\mathbf{I} - \mathbf{H}^{-1} \mathbf{U} \mathbf{G}] = \boldsymbol{\alpha}^T.$$

It is helpful to recall the definitions of the terms in the spillover matrix. The matrix $\mathbf{U}$ is diagonal with entries

$$U_{jj} = \sum_{s \in \mathcal{S}} P'_s(Y^*) u_j(\tau_j^*(s)).$$

We showed in [Lemma 4](#) that any agent receives a positive payment under an optimal contract $\boldsymbol{\tau}^*$ only at outcomes where $P'_s(Y^*) > 0$. An implication of this is that each diagonal entry in $\mathbf{U}$ is positive. The matrix $\mathbf{G}$ is non-negative by Part (c) of [Assumption 8](#). The matrix $\mathbf{H}$ is diagonal with entries

$$H_{jj} = C''_j(a_j^*).$$

Since we assume cost functions are strictly convex, these diagonal entries are positive. It follows that $\mathbf{H}^{-1} \mathbf{U} \mathbf{G}$ is non-negative, so by the Perron-Frobenius theorem this matrix has a right eigenvector $\mathbf{p}$ with non-negative real entries and a positive real eigenvalue. Then

$$\mathbf{c}^T [\mathbf{I} - \mathbf{H}^{-1} \mathbf{U} \mathbf{G}] \mathbf{p} = \boldsymbol{\alpha}^T \mathbf{p},$$

which can be simplified to

$$(1 - \rho) \mathbf{c}^T \mathbf{p} = \boldsymbol{\alpha}^T \mathbf{p}. \tag{A.37}$$

Now, suppose that $\mathbf{H}^{-1} \mathbf{U} \mathbf{G}$ has spectral radius $\rho \geq 1$. By assumption, the team performance $Y(\cdot)$ is strictly increasing in each of its arguments. It follows that the right-hand side of (A.37) is positive. If the spillover matrix has spectral radius equal to 1, the left-hand is 0 while the right-hand is positive. Thus, we cannot have spectral radius 1.

We must show we cannot have a spectral radius $\rho > 1$. Since $1 - \rho$ is negative, this implies $\mathbf{c}^T \mathbf{p}$ is negative. We will construct a direction $\mathbf{t}$ such that when the optimal contract $\boldsymbol{\tau}^*$ is perturbed in direction $\mathbf{t}$, the agents' individual incentives move in direction $-\mathbf{p}$. The resulting change in the equilibrium team performance is proportional to $-\mathbf{c}^T \mathbf{p}$, which is positive. This will contradict the optimality of contract $\boldsymbol{\tau}^*$.

We now state a lemma that characterizes the change in team performance when the contract is perturbed in *some* direction $\mathbf{t}$, where $t_i(s)$ specifies the change in payments to agent i in each outcome s . (Payments only to agents receiving a positive payment are perturbed.) The result below generalizes [Lemma 1](#) to any arbitrary direction.

Lemma 11. *The change in equilibrium team performance as payments to agents are perturbed in direction $\mathbf{t}$ is*

$$dY(\mathbf{t}) = l \sum_i \alpha_i c_i \left(\sum_{s \in \mathcal{S}} P'_s(Y^*) t_i(s) \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)} \right),$$

for some constant l .

The proof of the result above follows the same approach as the proof of [Lemma 9](#) so we omit it for brevity. We utilize this expression of change in team performance to analyze the principal's first-order condition. The derivative of the principal's objective is:

$$dY(\mathbf{t}) \underbrace{\sum_{s \in \mathcal{S}} \left(v_s - \sum_{i=1}^n \tau_i^*(s) \right) P'_s(Y^*)}_{D} - \sum_i \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s)$$

Substituting the expression in [Lemma 11](#), we get

$$lD \sum_i \alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y^*) t_i(s) \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)} - \sum_i \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s). \quad (\text{A.38})$$

We will show there exists a direction of perturbation $\mathbf{t}$ satisfying the following properties:

- Every element of eigenvector $\mathbf{p}$ satisfies

$$p_i = -\alpha_i \sum_{s \in \mathcal{S}} P'_s(Y^*) t_i(s) \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)},$$

- $t_i(s) \leq 0$ for any agent i and outcome s , and
- $t_i(s) < 0$ only if $\tau_i^*(s) > 0$.

For each agent i , choose an outcome s_i where he receives a positive payment. (Recall we have already restricted to the set of agents who receive a positive payment at some outcome, so this is possible.) At such an outcome s_i , the probability $P'_{s_i}(Y^*) > 0$. For outcome s_i , define

$$t_i(s_i) := -\frac{p_i C''_i(a_i^*)}{\alpha_i P'_{s_i}(Y^*) u'_i(\tau_i^*(s_i))}.$$

We have $t_i(s_i) \leq 0$ because $\alpha_i > 0$ and the entries of $\mathbf{p}$ are non-negative. For any other outcomes $s \in \mathcal{S} \setminus s_i$, define $t_i(s) := 0$.

Substituting in (A.38), the derivative of the principal's objective in direction $\mathbf{t}$ is

$$-lD\mathbf{c}^T \mathbf{p} - \sum_i P_{s_i}(Y^*) t_i(s_i). \quad (\text{A.39})$$

The rest of the proof establishes that the expression above is positive. First, we show that the quantity $lD > 0$. By Part (a) of Assumption 8, at the optimal contract $\boldsymbol{\tau}^*$, there exists some agent i satisfying the following conditions on payments and centrality: $\tau_i^*(s) > 0$ for some outcome s and $c_i > 0$. It must be that the principal's first-order condition (see proof of Theorem 1) applied to agent i is satisfied with equality, that is,

$$lD \alpha_i c_i P'_s(Y^*) \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)} = P_s(Y^*).$$

Moreover, since $c_i > 0$ we must have $lD > 0$. Combining this with $\mathbf{c}^T \mathbf{p} < 0$ it follows that

the first term $-lD\mathbf{c}^T\mathbf{p} > 0$. The inequality

$$-lD\mathbf{c}^T\mathbf{p} - \sum_i P_{s_i}(Y^*)t_i(s_i) > 0 \quad (\text{A.40})$$

follows from noting that $t_i(s_i) \leq 0$ for every agent i .

Since $\tau_i^*(s) > 0$ whenever $t_i(s) \neq 0$, a sufficiently small perturbation in direction $\mathbf{t}$ is feasible. So $\boldsymbol{\tau}^*$ cannot be optimal, which gives a contradiction. We conclude that at the optimal contract $\boldsymbol{\tau}^*$, the spillover matrix $\mathbf{H}^{-1}\mathbf{U}\mathbf{G}$ has spectral radius $\rho < 1$. $\square$

To characterize whether an agent receives a positive payment, it is useful to know whether the agent's centrality is positive. We can apply [Lemma 10](#) to show that the centrality of each agent receiving a positive payment is strictly positive. To see this, recall that centralities are defined by

$$\mathbf{c}^T = \boldsymbol{\alpha}^T [\mathbf{I} - \mathbf{H}^{-1}\mathbf{U}\mathbf{G}]^{-1}.$$

Since at an optimal contract the spectral radius of $\mathbf{H}^{-1}\mathbf{U}\mathbf{G}$ is strictly smaller than 1, we can expand the right-hand side as a power series:

$$\mathbf{c}^T = \boldsymbol{\alpha}^T \sum_{k=0}^{\infty} (\mathbf{H}^{-1}\mathbf{U}\mathbf{G})^k.$$

The spillover matrix $\mathbf{H}^{-1}\mathbf{U}\mathbf{G}$ is non-negative (see the proof of [Lemma 10](#)) while each entry of $\boldsymbol{\alpha}$ is strictly positive. We conclude that the centrality c_i of each agent receiving a positive payment is strictly positive.

Unfortunately, this does not allow us to conclude that *all* agents receive a payment. Suppose some agent received payment zero under $\boldsymbol{\tau}^*$. The Inada condition guarantees a small payment to that agent under suitable outcomes would provide a large incentive to work. But whether this incentive helps the principal depends on the sign of that agent's centrality. We will now show that agents that do not receive a payment at the optimal contract have a strictly positive centrality as well.

Our definition of centrality in [Section 1.3.1](#) focused on agents that receive a payment. We will extend the definition to *all* agents. To do so, we extend various other definitions to allow entries for every agent. Define the vector $\tilde{\alpha}$ by

$$\tilde{\alpha} := \nabla Y(\mathbf{a}^*).$$

Define the matrix $\tilde{\mathbf{U}} \in \mathbb{R}^{n \times n}$ to be diagonal with entries

$$\tilde{U}_{jj} := \sum_{s \in \mathcal{S}} P'_s(Y^*) u_j(\tau_j^*(s)).$$

For any agent that does not receive positive payments under any outcome, the diagonal entry is 0. Define $\tilde{\mathbf{G}} \in \mathbb{R}^{n \times n}$ to be the transpose of the Hessian matrix, i.e.,

$$\tilde{G}_{jk} := \frac{\partial^2 Y}{\partial a_k \partial a_j}.$$

Observe that the Hessian $\mathbf{G}$ defined for agents that receive a payment is a submatrix of $\tilde{\mathbf{G}}$. Define the matrix $\tilde{\mathbf{H}} \in \mathbb{R}^{n \times n}$ to be diagonal with entries

$$\tilde{H}_{jj} := C''_j(a_j^*).$$

We can now define all agents' centralities given the optimal contract $\boldsymbol{\tau}^*$ by

$$\tilde{\mathbf{c}}^T := \tilde{\alpha}^T \left[\mathbf{I} - \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{U}} \tilde{\mathbf{G}} \right]^{-1}. \quad (\text{A.41})$$

Lemma 12. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract. For any agent i , the centrality $\tilde{c}_i > 0$.*

Proof. We first verify for agents that receive a payment that their centrality defined in $\tilde{\mathbf{c}}$ is equal to their centrality as defined in $\mathbf{c}$. For ease of notation, let the spillover matrix on *all* agents $\tilde{\mathbf{S}} = \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{U}} \tilde{\mathbf{G}}$ and for those with a payment $\mathbf{S} = \mathbf{H}^{-1} \mathbf{U} \mathbf{G}$.

We will show that $\tilde{\mathbf{S}}$ and $\mathbf{S}$ have the same non-zero eigenvalues. Consequently, they have the same spectral radius. Suppose μ is an eigenvalue of $\mathbf{S}$ with corresponding eigenvector $\mathbf{v}$. Then, μ is also an eigenvalue of $\tilde{\mathbf{S}}$. To see this, suppose (without loss of generality)

agents with a payment are labeled $\{1, \dots, k\}$. Rows $(k+1)$ onwards in $\tilde{\mathbf{S}}$ have all zeros. The matrix $\mathbf{S}$ is the top-left $(k \times k)$ dimensional submatrix of $\tilde{\mathbf{S}}$. We can define a n -dimensional vector $\tilde{\mathbf{v}}$ as $\tilde{v}_i := v_i$ when $i \leq k$ and $\tilde{v}_i := 0$ when $i > k$. It is straightforward to see $\tilde{\mathbf{v}}$ is an eigenvector of $\tilde{\mathbf{S}}$ with eigenvalue μ . We will now prove the other direction. Suppose $\tilde{\mu}$ is a non-zero eigenvalue of $\tilde{\mathbf{S}}$ with corresponding eigenvector $\tilde{\mathbf{v}}$. Since rows $(k+1)$ onwards in $\tilde{\mathbf{S}}$ have all zeros, it must be that $\tilde{v}_i = 0$ for components $i \geq (k+1)$. But this implies $\mathbf{v}$, corresponding to the first k components of $\tilde{\mathbf{v}}$, is an eigenvector of $\mathbf{S}$ with eigenvalue $\tilde{\mu}$. Applying [Lemma 10](#) tells us the spectral radius of $\tilde{\mathbf{S}}$ is strictly smaller than 1.

So we have the following power series expansion of [\(A.41\)](#):

$$\tilde{\mathbf{c}}^T = \tilde{\boldsymbol{\alpha}}^T \sum_{\ell=0}^{\infty} \tilde{\mathbf{S}}^{\ell}. \quad (\text{A.42})$$

We can write for any $\ell \geq 1$

$$\tilde{\mathbf{S}}^{\ell} = \begin{bmatrix} \mathbf{S}^{\ell} & \mathbf{J}_{\ell} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{J}_{\ell}$ is a matrix that does not contribute to the centrality. Substituting in [\(A.42\)](#) and noting that the first k elements of $\tilde{\boldsymbol{\alpha}}$ are just the vector $\boldsymbol{\alpha}$, we get $\tilde{c}_i = c_i$ for agents that receive a payment. The centrality $\tilde{c}_i$ for an agent without a payment will affect overall team performance when their payment at a particular outcome is perturbed.

All that remains to show is that $\tilde{c}_i > 0$ for every agent. This follows from [\(A.42\)](#): $\tilde{\mathbf{S}}$ is non-negative and every element of $\tilde{\boldsymbol{\alpha}}$ is strictly positive. $\square$

We now complete the proof of the proposition.

Proof of [Proposition 18](#). The proof involves analyzing the derivative of the principal's objective with respect to payments made to the agents. Consider any agent i . As shown in the proof of [Theorem 1](#), the derivative of the principal's objective with respect to $\tau_i(s)$ is

$$lD\alpha_i c_i \cdot \frac{u'_i(\tau_i^*(s))}{C''_i(a_i^*)} \cdot P'_s(Y^*) - P_s(Y^*).$$

Forward direction: Let $\mathcal{S}_i^*$ be the set of outcomes at which an agent i receives a positive payment. Then

$$P'_s(Y^*) > 0 \text{ for all } s \in \mathcal{S}_i^*.$$

The statement of the forward direction is exactly [Lemma 4](#). Note that the arguments to prove the lemma did not require an Inada condition.

Backward direction: Any agent i receives a strictly positive payment at all outcomes where $P'_s(Y^*) > 0$.

By Part (a) of [Assumption 8](#), at the optimal contract τ^* , there exists some agent i receiving a positive payment at some outcome s and has centrality $c_i > 0$. Note that from the *forward direction*, we must have $P'_s(Y^*) > 0$. The principal's first-order condition (see proof of [Theorem 1](#)), applied to agent i is

$$lD\alpha_i c_i \cdot \frac{u'_i(\tau_i^*(s))}{C_i'''(a_i^*)} \cdot P'_s(Y^*) - P_s(Y^*) = 0.$$

This implies $lD > 0$, because $c_i > 0$. For any other agent j , recall from the proof of [Theorem 1](#) that the derivative of the principal's objective in $\tau_j(s)$, is given by the expression

$$lD\alpha_j c_j \cdot \frac{u'_j(\tau_j^*(s))}{C_j'''(a_j^*)} \cdot P'_s(Y^*) - P_s(Y^*).$$

This expression was stated for any agent receiving a positive payment at some outcome, but also holds for agents receiving a zero payment at all outcomes as long as the perturbation is made at an outcome s at which $P'_s(Y^*) > 0$.⁴ It is straightforward to verify that c_j is exactly

⁴Recall that [Lemma 1](#) was defined for any agent receiving a positive payment at some outcome. We show that the result also holds for agents receiving zero payments at all outcomes, when the perturbation in payments is made in an outcome where $P'_s(Y^*) > 0$. For any agent i taking action $a_i^* > 0$, the first-order conditions at equilibrium imply that a_i^* solves

$$C'_i(a_i) = \left(\sum_{s' \in \mathcal{S}} P'_{s'}(Y) u_i(\tau_i(s')) \right) \frac{\partial Y}{\partial a_i}.$$

We show the equation must also hold if $a_i^* = 0$. To see this, recall that $a_i^* = 0$ if and only if $\tau_i(s) = 0$ at all

the term that appears in the centrality vector $\tilde{\mathbf{c}}$ defined in the proof of [Lemma 12](#). By the Inada condition on the marginal utility function, the observation that $lDc_j > 0$ (which holds because $c_j > 0$ as shown in [Lemma 12](#)), and the fact that $P'_s(Y^*) > 0$

$$\lim_{\tau_j^*(s) \rightarrow 0} lD\alpha_j c_j \cdot \frac{u'_j(\tau_j^*(s))}{C''_j(a_j^*)} P'_s(Y^*) - P_s(Y^*) > 0.$$

Thus, we cannot have $\tau_j^*(s) = 0$ at an optimal contract. $\square$

A.5 Optimal equity pay

The contracts in our main model are finely tailored to individual outcomes (see [Corollary 2](#)). In practice, such contracts may be difficult to implement, and firms often use simple compensation schemes. Our results can be adapted to characterize optimal contracts within a restricted class. This section provides an illustration by analyzing one widely used incentive scheme: equity pay. Note that in simple success-or-failure environments, all optimal contracts feature equity pay, but in general the optimal equity contract need not match the optimal unrestricted contract.

An equity pay contract pays each agent a fixed share $\tau_i v_s$ of the surplus v_s produced by the team. For a given equity contract $\boldsymbol{\tau}$, the expected payoff to the principal is

$$\left(1 - \sum_{i \in N} \tau_i\right) \sum_{s \in \mathcal{S}} v_s P_s(Y).$$

outcomes s . Consider the first order-condition when $a_i^* = 0$ and $\tau_i(s) = 0$ for all s :

$$C'_i(0) = \left(\sum_{s' \in \mathcal{S}} P'_{s'}(Y) u_i(0) \right) \frac{\partial Y}{\partial a_i}.$$

The left-hand side is zero because $C'(0) = 0$. The right-hand side is zero since $\sum_{s' \in \mathcal{S}} P'_{s'}(Y) = 0$. So the first-order condition holds in this case as well. It follows that the first-order condition binds when payments are perturbed for such an agent at an outcome where $P'_s(Y^*) > 0$.

The expected payoff to agent i from an equity share τ_i is

$$\mathcal{U}_i = \sum_{s \in \mathcal{S}} u_i(\tau_i v_s) P_s(Y) - C_i(a_i).$$

The result below characterizes an optimal equity contract $\boldsymbol{\tau}^*$. We maintain [Assumption 1](#), which now states that there is a neighborhood of $\boldsymbol{\tau}^*$ in the space of equity contracts where $\boldsymbol{a}^*(\boldsymbol{\tau})$ is continuously differentiable.

Proposition 19. *Suppose $\boldsymbol{\tau}^*$ is an optimal equity contract and Y^* is the induced team performance. There exists a constant λ such that for any agent i receiving a positive equity payment, we have*

$$\alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y^*) v_s \cdot \frac{u'_i(\tau_i^* v_s)}{C''_i(a_i^*)} = \lambda.$$

The proof, which we provide below, follows a similar approach to the proof of [Theorem 1](#). It involves analyzing the effect of perturbations to equity payments on the principal's objective. Perturbations in the equity payment of an agent affect payments at all outcomes. The direct effect of increasing τ_i on i 's action is proportional to the change in marginal expected utility from payments, which is given by the expression

$$\alpha_i \sum_{s \in \mathcal{S}} P'_s(Y^*) v_s \cdot \frac{u'_i(\tau_i^* v_s)}{C''_i(a_i^*)}.$$

The summation captures the total direct effect of increasing an agent's equity on team performance by aggregating across outcomes, and multiplying by c_i includes indirect effects. At an optimal equity contract, the effect of perturbing equity payments on total team performance must be the same for all agents with positive equity.

In general, the balance condition in [Proposition 19](#) characterizing optimal equity contracts does not match the condition in [Section 1.3](#) characterizing optimal contracts. An optimal contract fine-tunes payments at each outcome, incentivizing agents to exert optimal effort levels. Equity pay imposes a particular linear relationship between the payments for different outcomes that may be practically convenient but sacrifices some incentive power. We

note the contrast with Dai and Toikka [2022], building on Carroll [2015], which finds that linear contracts are optimal for a principal designing team incentives that must be robust to uncertainty about the environment.

Proof of Proposition 19. We begin with a lemma, which adapts Lemma 1.

Lemma 13. *Suppose τ^* is an optimal equity contract with corresponding equilibrium actions $\mathbf{a}^*$ and team performance Y^* . For any agent i , the derivative of team performance in τ_i , evaluated at τ^* , is*

$$\frac{dY}{d\tau_i} = l\alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y^*) v_s \cdot \frac{u'_i(\tau_i^* v_s)}{C''_i(a_i^*)},$$

where l is independent of i and s .

Proof. The steps taken in this proof are exactly the same as those taken in the proof of Lemma 1. We analyze the change in team performance as the equity transfer to an agent is perturbed. Consider an equity payment scheme τ^* and any agent i . Consider marginally increasing τ_i . The change induced by this perturbation is

$$\frac{\partial Y}{\partial \tau_i} = \nabla Y(\mathbf{a}^*)^T \cdot \frac{\partial \mathbf{a}^*}{\partial \tau_i}, \quad (\text{A.43})$$

where $\mathbf{a}^*$ is the equilibrium action profile for the contract τ . The substance of the proof is analyzing the second term on the right-hand side of (A.43).

As in Lemma 1, it is without loss to analyze the change in the action of agent i and actions of agents j that take strictly positive actions in profile $\mathbf{a}^*$. The analysis from here on focuses on such agents, overloading notation to represent the actions of these agents by $\mathbf{a}^*$.

We will show that the change in equilibrium actions $\mathbf{a}^*$ as the equity τ_i increases is

$$\frac{\partial \mathbf{a}^*}{\partial \tau_i} = [\mathbf{I} - \mathbf{H}^{-1} \mathbf{U} \mathbf{G}]^{-1} \mathbf{H}^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} \sum_{s \in \mathcal{S}} P'_s(Y) v_s u'_i(\tau_i v_s) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i} [\mathbf{H} - \mathbf{U} \mathbf{G}]^{-1} \mathbf{d}, \quad (\text{A.44})$$

where $\mathbf{d}$ is a vector that depends on the curvature of the probability of outcome $P_s(\cdot)$.

Consider the equilibrium action profile $\mathbf{a}^*$. For an agent j , the first-order conditions imply a_j^* must solve the equation

$$C'_j(a_j) = \left(\sum_{s \in \mathcal{S}} P'_s(Y) u_j(\tau_j v_s) \right) \frac{\partial Y}{\partial a_j}. \quad (\text{A.45})$$

To arrive at (A.44), let us implicitly differentiate (A.45) with respect to τ_i . For all $j \neq i$,

$$C''_j(a_j^*) \frac{\partial a_j^*}{\partial \tau_i} = \left(\sum_{s \in \mathcal{S}} P'_s(Y) u_j(\tau_j v_s) \right) \left(\sum_{k=1}^n \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \tau_i} \right) \quad (\text{A.46})$$

$$+ \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \tau_i} \cdot \sum_{s \in \mathcal{S}} P''_s(Y) u_j(\tau_j v_s). \quad (\text{A.47})$$

Similarly for $j = i$,

$$C''_j(a_j^*) \frac{\partial a_j^*}{\partial \tau_i} = \left(\sum_{s \in \mathcal{S}} P'_s(Y) u_j(\tau_j v_s) \right) \left(\sum_{k=1}^n \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \tau_i} \right) \quad (\text{A.48})$$

$$+ \frac{\partial Y}{\partial a_j} \sum_{s \in \mathcal{S}} P'_s(Y) v_s u'_j(\tau_j v_s) + \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \tau_i} \sum_{s \in \mathcal{S}} P''_s(Y) u_j(\tau_j v_s). \quad (\text{A.49})$$

We can combine (A.46) and (A.48) in vector form:

$$\frac{\partial \mathbf{a}^*}{\partial \tau_i} = [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} \sum_{s \in \mathcal{S}} P'_s(Y) v_s u'_i(\tau_i v_s) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i} [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d},$$

where $\mathbf{d}$ is a vector with $d_j = \frac{\partial Y}{\partial a_j} \sum_{s \in \mathcal{S}} P''_s(Y) u_j(\tau_j v_s)$. The expression in (A.44) follows.

Substituting (A.44) into (A.43), the change in team performance as the equity payment τ_i

increases is

$$\begin{aligned} \frac{\partial Y}{\partial \tau_i} &= \nabla Y(\mathbf{a}^*)^T [\mathbf{I} - \mathbf{H}^{-1} \mathbf{U} \mathbf{G}]^{-1} \mathbf{H}^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} \sum_{s \in \mathcal{S}} P'_s(Y) v_s u'_i(\tau_i v_s) \\ \mathbf{0} \end{bmatrix} \\ &\quad + \frac{\partial Y}{\partial \tau_i} \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U} \mathbf{G}]^{-1} \mathbf{d}. \end{aligned}$$

Applying the definitions of α_i and c_i , we obtain

$$\frac{\partial Y}{\partial \tau_i} = \alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y) v_s \cdot \frac{u'_i(\tau_i^* v_s)}{C''_i(a_i^*)} + \frac{\partial Y}{\partial \tau_i} \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U} \mathbf{G}]^{-1} \mathbf{d}.$$

Rearranging,

$$\frac{\partial Y}{\partial \tau_i} = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U} \mathbf{G}]^{-1} \mathbf{d}} \cdot \alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y) v_s \cdot \frac{u'_i(\tau_i^* v_s)}{C''_i(a_i^*)}.$$

Setting $l = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U} \mathbf{G}]^{-1} \mathbf{d}}$ and observing l does not depend on i , we obtain the desired result. $\square$

The expected payoff for the principal under equity payment $\boldsymbol{\tau}$ and corresponding equilibrium actions $\mathbf{a}^*$ is

$$\left(1 - \sum_{i \in N} \tau_i\right) \sum_{s \in \mathcal{S}} v_s P_s(Y(\mathbf{a}^*)).$$

Suppose $\boldsymbol{\tau}^*$ is an optimal equity contract inducing equilibrium $\mathbf{a}^*(\boldsymbol{\tau}^*)$ with team performance Y^* . Consider agent i such that $\tau_i^* > 0$. Then the first-order condition for τ_i^* implies that

$$\frac{dY}{d\tau_i} \cdot \underbrace{\left(1 - \sum_{i \in N} \tau_i^*\right) \sum_{s \in \mathcal{S}} v_s P'_s(Y^*)}_D = \sum_{s \in \mathcal{S}} v_s P_s(Y^*).$$

The left-hand side is the benefit from increasing τ_i^* while the right-hand side is the expected additional transfer required. Since each outcome occurs with non-zero probability,

the summation labeled D is nonzero.

Substituting [Lemma 13](#) in the above equation, we obtain

$$\begin{aligned} l\alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y^*) v_s \cdot \frac{u'_i(\tau_i^* v_s)}{C''_i(a_i^*)} &= \frac{\sum_{s \in \mathcal{S}} v_s P_s(Y^*)}{D}, \\ \iff \alpha_i c_i \sum_{s \in \mathcal{S}} P'_s(Y^*) v_s \cdot \frac{u'_i(\tau_i^* v_s)}{C''_i(a_i^*)} &= \lambda, \end{aligned}$$

where $\lambda = \sum_{s \in \mathcal{S}} v_s P_s(Y^*) / (lD)$. Observing that λ is independent of i and the outcome s , the statement of the result follows. $\square$

Appendix B

Appendix for Equity Pay In Networked Teams

B.1 Proofs

We next state and prove a lemma, discussed in [Section 2.3](#), which will be used in subsequent proofs.

Lemma 14. *An allocation τ is optimal among those with a given sum of shares $\sum_{i \in N} \sigma_i = s$ if and only if it solves*

$$\begin{aligned} \max_{\tau} \quad & c \\ \text{subject to} \quad & (G\sigma)_i = c \text{ whenever } \sigma_i > 0. \end{aligned}$$

In particular, the active sets at optimal allocations are the same as the sets of non-negative indices under solutions to this optimization problem.

Proof. By [Proposition 2](#), there exists a constant c such that for all agents that get a strictly positive equity at the optimal solution, $(G\sigma^*)_i = c$. At any solution which satisfies the

balanced equity condition and allocates a fraction s of shares to agents, the team performance

$$Y^* = \mathbf{1}^T \mathbf{a}^* + \frac{\beta}{2} (\mathbf{a}^*)^T \mathbf{G} \mathbf{a}^*$$

can be rewritten as

$$Y^* = \left(\frac{P'(Y^*)}{1 - \beta P'(Y^*)c} + \frac{\beta P'(Y^*)^2 c}{2(1 - \beta P'(Y^*)c)^2} \right) s. \quad (\text{B.1})$$

We will conclude from the above expression that team performance is increasing in c . For a given c , the team performance $Y^*(c)$ is the solution to $f(y, c) = y$, where we define

$$f(y, c) := \left(\frac{P'(y)}{1 - \beta P'(y)c} + \frac{\beta P'(y)^2 c}{2(1 - \beta P'(y)c)^2} \right) s.$$

By assumption $P(\cdot)$ is concave and twice differentiable, so $f(y, c)$ is decreasing in y . Since we have also assumed $P(\cdot)$ is strictly increasing, we have $\frac{\partial f}{\partial c} > 0$ for all y and thus $Y^*(c)$ is increasing in c . $\square$

Proof of Proposition 5. By Lemma 14, the optimal active set is determined by an optimization problem that is independent of β . So if the active set for the allocation $\boldsymbol{\tau}_0^*$ is optimal for complementarity parameter β_0 , the same active set is optimal for complementarity parameter β_1 . Call this active set $\tilde{\mathbf{G}}$.

By Proposition 2, the optimal allocation with this active set is characterized by $\boldsymbol{\tau}_0^* = \tilde{\mathbf{G}}^{-1} \mathbf{1} c_0$ for some constant $c_0 > 0$ when the complementarity parameter is β_0 . Similarly, the allocation with this active set is characterized by $\boldsymbol{\tau}_1^* = \tilde{\mathbf{G}}^{-1} \mathbf{1} c_1$ for some constant $c_1 > 0$ when the complementarity parameter is β_1 . This implies the claim with $k = c_1/c_0$. $\square$

Proof of Proposition 6. We consider the success probability objective as the argument is essentially the same for both objectives. By Lemma 14, any optimal allocation maximizes $(G\boldsymbol{\tau})_i$ for active agents i among allocations $\boldsymbol{\tau}$ satisfying the balanced equity condition.

Let $\bar{g} = \max_{i,j} G_{ij}$ and choose i and j such that the link between i and j obtains this

maximum weight. Setting $\sigma_i = \sigma_j = \frac{1}{2}$ and all other $\sigma_k = 0$ gives $(G\boldsymbol{\tau})_i = (G\boldsymbol{\tau})_j = \bar{g}/2$. We now show this value cannot be obtained with an active set with diameter greater than 2.

Suppose there is an optimal allocation $\boldsymbol{\tau}^*$ with an active set A^* with diameter greater than 2. Choose active agents i and j such that the distance between i and j is at least 2. The subsets $\{i\}, \{j\}, N(i) \cap A^*$, and $N(j) \cap A^*$ of the active set are all disjoint.

The balanced equity condition implies that $(G\boldsymbol{\tau}^*)_i = (G\boldsymbol{\tau}^*)_j = c$ for some constant c , and we have

$$\begin{aligned} 2c &= (G\boldsymbol{\tau}^*)_i + (G\boldsymbol{\tau}^*)_j \\ &\leq \bar{g} \sum_{k \in N(i) \cup N(j)} \sigma_k^* \\ &< \bar{g}, \end{aligned}$$

where the last inequality holds because $\sigma_i^* > 0$ so $\sum_{k \in N(i) \cup N(j)} \sigma_k^* < 1$. Since we showed we can obtain a value of $c = \bar{g}/2$, this contradicts the optimality of $\boldsymbol{\tau}^*$. $\square$

Proof of Theorem 3. We consider the success probability objective as the argument is essentially the same for both objectives. By Lemma 14, any optimal allocation maximizes the constant $c = (G\boldsymbol{\tau})_i$ for active agents i among allocations $\boldsymbol{\tau}$ satisfying the balanced equity condition.

Let the size of the maximum clique in the network be $\bar{k}$. Proposition 2 implies that the optimal allocation with active set a clique of size k gives all active agents equal shares $\frac{1}{k}$. Under this allocation, the balanced equity condition holds with constant $c = \frac{\bar{k}-1}{k}$.

Suppose an allocation $\boldsymbol{\tau}$ satisfies the balanced neighborhood equity condition with constant $c > \frac{\bar{k}-1}{\bar{k}}$. We will show the active set under this allocation must contain a clique of size at least $\bar{k} + 1$, which contradicts our assumption that the size of the maximum clique is $\bar{k}$.

Define

$$k^* = \arg \max_{k \in \mathbb{Z}} \left\{ c - \left(\frac{k-1}{k} \right) > 0 \right\}.$$

We will show that the active set under τ contains a clique of size $k^* + 1 > \bar{k}$. Call the set of vertices of this active set by A^* . First observe that the equity that each agent gets is at most $(1 - c)$. This is because each agent's neighbors receive equity shares summing to c and the total of all equity shares is 1.

We will define a sequence of agents $i^0, \dots, i^{k^*}$ inductively such that $i^0, i^1, \dots, i^k$ is a clique for all k . Fix some i^0 in the active set and define $N_S(i^0) := N(i^0)$. Given $i^0, i^1, \dots, i^k$ for any $k < k^*$, we define

$$N_S(i^k) := \bigcap_{l=0}^k N(i^l).$$

Given $i^0, \dots, i^k$ with $0 \leq k < k^*$, we want to choose i^{k+1} to be an arbitrary agent in $N_S(i^k)$. To do so, we must show $N_S(i^k)$ is non-empty.

We will prove that the total equity in $N_S(i^k)$ is at least $(k + 1)c - k$, i.e.,

$$\sum_{i \in N_S(i^k)} \sigma_i \geq (k + 1)c - k.$$

We show this by induction on k . The base case $k = 0$ holds by the balanced neighborhood equity condition.

The inductive hypothesis is

$$\sum_{i \in N_S(i^{k-1})} \sigma_i \geq kc - (k - 1).$$

This implies

$$\sum_{i \in N(i^0) \setminus N_S(i^{k-1})} \sigma_i \leq (k - 1)(1 - c) \tag{B.2}$$

since $\sum_{i \in N(i^0)} \sigma_i = c$.

Since i^k is active, we also have $\sum_{i \in N(i^k)} \sigma_i = c$. We can decompose the equity in this neighborhood, potentially along with additional agents' shares, as

$$\sum_{i \in A^* \setminus N(i^0)} \sigma_i + \sum_{i \in N(i^0) \setminus N_S(i^{k-1})} \sigma_i + \sum_{i \in N_S(i^k)} \sigma_i \geq c.$$

By the balanced neighborhood equity condition for agent i^0 and (B.2), this implies

$$\sum_{i \in N_S(i^k)} \sigma_i \geq (k+1)c - k,$$

completing the induction. Since $c > \frac{k^*-1}{k^*}$, this implies that $N_S(i^k)$ is non-empty for each $k \in \{1, \dots, k^* - 1\}$. So we can construct $i^0, \dots, i^{k^*}$ as described above.

By construction, the subnetwork $\{i^0, \dots, i^{k^*}\}$ is a clique of size $k^* + 1$. Since we have assumed the maximum clique has size $\bar{k}$, this contradicts the existence of an allocation τ satisfying the balanced neighborhood equity condition with constant $c > \frac{\bar{k}-1}{\bar{k}}$. Thus the maximum clique must be an optimal allocation. $\square$

Proof of Proposition 7. Proposition 2 tells us that, for all agents such that $\tau_i^* > 0$, we have

$$\tau^* = c\tilde{\mathbf{G}}^{-1}\mathbf{1}.$$

We will use the matrix calculus expression

$$\frac{\partial \mathbf{G}(t)^{-1}}{\partial t} = -\mathbf{G}(t)^{-1} \frac{\partial \mathbf{G}(t)}{\partial t} \mathbf{G}(t)^{-1}.$$

Taking the derivative with respect to G_{jk} , we have that

$$\frac{\partial \tau^*}{\partial G_{jk}} = -c\tilde{\mathbf{G}}^{-1} \frac{\partial \tilde{\mathbf{G}}}{\partial G_{jk}} \tilde{\mathbf{G}}^{-1}\mathbf{1} + \frac{\partial c}{\partial G_{jk}} \tilde{\mathbf{G}}^{-1}\mathbf{1}.$$

Analyzing the i^{th} element in this vector gives

$$\frac{\partial \tau_i^*}{\partial G_{jk}} = -c(\tilde{\mathbf{G}}^{-1}\mathbf{1})_j(\tilde{\mathbf{G}}^{-1})_{ik} - c(\tilde{\mathbf{G}}^{-1}\mathbf{1})_k(\tilde{\mathbf{G}}^{-1})_{ij} + \frac{\partial c}{\partial G_{jk}} \cdot (\tilde{\mathbf{G}}^{-1}\mathbf{1})_i.$$

The result follows from $\sigma_i^* = c(\tilde{\mathbf{G}}^{-1}\mathbf{1})_i$ and the analogous expressions with indices j and k . $\square$

Proof of Proposition 8. We want to calculate the derivative of the team performance Y^*

under the optimal allocation as G_{ij} increases. By the envelope theorem, we can calculate this derivative by holding fixed the allocation $\boldsymbol{\tau}^*$. To do so, we calculate the derivative of the equilibrium team performance Y^* for a given allocation $\boldsymbol{\sigma}$ as G_{ij} increases. We will then substitute $\boldsymbol{\sigma} = \boldsymbol{\tau}^*$.

Letting $\mathbf{a}^*$ be the equilibrium action profile under the allocation $\boldsymbol{\sigma}$, we calculate

$$\begin{aligned} \frac{\partial Y}{\partial G_{ij}} &= \frac{\partial \mathbf{1}^T \mathbf{a}^* + \frac{\beta}{2} (\mathbf{a}^*)^T \mathbf{G} \mathbf{a}^*}{\partial G_{ij}}, \\ &= \mathbf{1}^T \frac{\partial \mathbf{a}^*}{\partial G_{ij}} + \beta (\mathbf{a}^*)^T \mathbf{G} \frac{\partial \mathbf{a}^*}{\partial G_{ij}} + \frac{\beta}{2} (\mathbf{a}^*)^T \frac{\partial \mathbf{G}}{\partial G_{ij}} \mathbf{a}^*, \\ &= [\mathbf{1}^T + \beta (\mathbf{a}^*)^T \mathbf{G}] \frac{\partial \mathbf{a}^*}{\partial G_{ij}} + \beta a_i^* a_j^*. \end{aligned}$$

The equilibrium action satisfies $\mathbf{a}^* = \beta P'(Y) \boldsymbol{\Sigma} \mathbf{G} \mathbf{a}^* + P'(Y) \boldsymbol{\Sigma} \mathbf{1}$. Thus, we can write

$$\begin{aligned} \frac{\partial \mathbf{a}^*}{\partial G_{ij}} &= \beta P'(Y) \boldsymbol{\Sigma} \frac{\partial \mathbf{G}}{\partial G_{ij}} \mathbf{a}^* + \beta P'(Y) \mathbf{T} \mathbf{G} \frac{\partial \mathbf{a}^*}{\partial G_{ij}} + (\beta \boldsymbol{\Sigma} \mathbf{G} \mathbf{a}^* + \boldsymbol{\Sigma} \mathbf{1}) \frac{\partial P'(Y)}{\partial G_{ij}}, \\ &= \beta \begin{bmatrix} 0 \\ \sigma_i a_j^* \\ 0 \\ \sigma_j a_i^* \\ 0 \end{bmatrix} P'(Y) + \beta P'(Y) \mathbf{T} \mathbf{G} \frac{\partial \mathbf{a}^*}{\partial G_{ij}} + (\beta \boldsymbol{\Sigma} \mathbf{G} \mathbf{a}^* + \boldsymbol{\Sigma} \mathbf{1}) P''(Y) \frac{\partial Y}{\partial G_{ij}}. \end{aligned}$$

where $\boldsymbol{\Sigma} \frac{\partial \mathbf{G}}{\partial G_{ij}} \mathbf{a}^*$ is a vector with the i^{th} element equal to $\sigma_i a_j^*$, the j^{th} element equal to $\sigma_j a_i^*$ and the rest of the elements equal to zero. Solving for $\frac{\partial \mathbf{a}^*}{\partial G_{ij}}$ gives

$$\frac{\partial \mathbf{a}^*}{\partial G_{ij}} = [\mathbf{I} - \beta P'(Y) \boldsymbol{\Sigma} \mathbf{G}]^{-1} \left(\beta P'(Y) \begin{bmatrix} 0 \\ \sigma_i a_j^* \\ 0 \\ \sigma_j a_i^* \\ 0 \end{bmatrix} + \boldsymbol{\Sigma} [\mathbf{1} + \beta \mathbf{G} \mathbf{a}^*] P''(Y) \frac{\partial Y}{\partial G_{ij}} \right).$$

Substituting into the expression for $\frac{\partial Y}{\partial G_{ij}}$ gives

$$\begin{aligned} & \frac{\partial Y}{\partial G_{ij}} \left[1 - (\mathbf{1} + \beta \mathbf{G} \mathbf{a}^*)^T [\mathbf{I} - \beta P'(Y) \mathbf{\Sigma} \mathbf{G}]^{-1} \mathbf{\Sigma} (\mathbf{1} + \beta \mathbf{G} \mathbf{a}^*) P''(Y) \right] \\ &= \beta P'(Y) [\mathbf{1}^T + \beta (\mathbf{a}^*)^T \mathbf{G}] [I - \beta P'(Y) \mathbf{T} \mathbf{G}]^{-1} \begin{bmatrix} 0 \\ \sigma_i a_j^* \\ 0 \\ \sigma_j a_i^* \\ 0 \end{bmatrix} + \beta a_i^* a_j^*. \end{aligned}$$

We now use the optimality of $\boldsymbol{\sigma}$, which implies the equality $\mathbf{a}^* = \boldsymbol{\tau}^* \frac{P'(Y)}{1 - \beta c P'(Y)}$ by [Proposition 2](#). Applying this, we obtain

$$\frac{\partial Y^*}{\partial G_{ij}} = \beta \sigma_i^* \sigma_j^* P'(Y^*)^2 \frac{\left(\frac{2}{(1 - \beta c P'(Y^*))^3} + \frac{1}{(1 - \beta c P'(Y^*))^2} \right)}{1 - \frac{P''(Y^*) \sum_i \sigma_i^*}{(1 - \beta c P'(Y^*))^3}}.$$

The right-hand side has the desired form. □

Proof of [Proposition 9](#). We can assume without loss of generality that all agents in $\mathbf{G}$ are active under $\boldsymbol{\tau}^*$ (by dropping any inactive agents from the network). Consider a feasible allocation $\boldsymbol{\tau}$ satisfying the balanced neighborhood equity condition $\mathbf{G} \boldsymbol{\sigma} = c \mathbf{1}$ and let $s = \sum_i \sigma_i \in [0, 1]$ be the sum of shares under this allocation. Such an allocation will exist for any $s \in [0, 1]$, as $\mathbf{G}$ is the optimal active set and thus $(\mathbf{G}^{-1} \mathbf{1})_i > 0$ for all i in $\mathbf{G}$. The balanced neighborhood equity condition implies that

$$c = \frac{s}{\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1}}.$$

Applying [\(B.1\)](#), we can write the residual profit for the principal under this allocation as

$$V(s, \beta) = \alpha^2 s(1 - s) \left(\frac{1}{1 - \beta \alpha \frac{s}{\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1}}} + \frac{\beta \alpha \frac{s}{\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1}}}{2 \left(1 - \beta \alpha \frac{s}{\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1}} \right)^2} \right).$$

So for a fixed β for which $\boldsymbol{\tau}^*$ is an optimal allocation, the sum of shares under this allocation solves the optimization problem

$$V^*(\beta) = \max_{s \in [0,1]} s(1-s) \left(\frac{1}{1 - \beta\alpha \frac{s}{\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1}}} + \frac{\beta\alpha \frac{s}{\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1}}}{2 \left(1 - \beta\alpha \frac{s}{\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1}}\right)^2} \right).$$

We will characterize the solution to this optimization problem. We define $k^* := \mathbf{1}^T \mathbf{G}^{-1} \mathbf{1}$ and claim that $\beta\alpha < k^*$. We must have $\beta \in \left(0, \frac{1}{\alpha \rho(\boldsymbol{\Sigma} \mathbf{G})}\right)$ by our assumption that equilibrium team performance is in $[0, \bar{Y}]$. Observe that $c = s/k^*$ is an eigenvalue for the matrix $\boldsymbol{\Sigma} \mathbf{G}$ for any $s \in [0, 1]$, with right eigenvector $\boldsymbol{\tau}$. Thus we have

$$\frac{s}{k^*} \leq \rho(\boldsymbol{\Sigma} \mathbf{G}) < \frac{1}{\beta\alpha}.$$

Choosing $s = 1$ then verifies the claim $\beta\alpha < k^*$.

We now return to the problem of maximizing $V(s, \beta)$. Taking the partial derivative with respect to s , we find

$$\frac{\partial V(s, \beta)}{\partial s} = \frac{k^* \alpha^2 (-(\beta\alpha)^2 s^3 + 3\beta\alpha k^* s^2 - 4(k^*)^2 s + 2(k^*)^2)}{2(k^* - \beta\alpha s)^3}.$$

It suffices to study the behavior of $V(s, \beta)$ when $s \in [0, 1]$. Define

$$p(s, \beta) := -(\beta\alpha)^2 s^3 + 3\beta\alpha k^* s^2 - 4(k^*)^2 s + 2(k^*)^2$$

to be the numerator of $V(s, \beta)$. The partial derivative of $p(s, \beta)$ with respect to s is

$$\frac{\partial p(s, \beta)}{\partial s} = -3(\beta\alpha)^2 s^2 + 6\beta\alpha k^* s - 4(k^*)^2 < -3(\beta\alpha s + k^*)^2.$$

Since the right-hand expression is strictly negative, the function $p(s, \beta)$ is strictly decreasing in $s \in [0, 1]$. Thus $p(\cdot, \beta)$ has only one real root for each β .

We claim that this root lies in $(\frac{1}{2}, 1)$. At $s = \frac{1}{2}$, we have

$$p\left(\frac{1}{2}, \beta\right) = \left(-(\beta\alpha)^2 \cdot \frac{1}{8} + 3\alpha\beta k^* \cdot \frac{1}{4}\right) > 0,$$

for any $\beta\alpha < k^*$. At $s = 1$, we have

$$p(1, \beta) = (k^* - \beta\alpha)(\beta\alpha - 2k^*) < 0,$$

for any $\beta\alpha < k^*$. This proves the claim.

For $s \in [0, 1]$, the denominator of $V(s, \beta)$ is strictly positive for any $\beta\alpha < k^*$. So for each β , the sum of shares s at the optimal allocation is characterized by $p(s, \beta) = 0$. We calculate

$$\frac{\partial p(s, \beta)}{\partial \beta} = 3\alpha k^* s^2 - 2\beta\alpha^2 s^3 = \alpha s^2 (3k^* - 2\beta\alpha s) > 0,$$

where the inequality holds for all $s \in (0, 1)$. Since $p(s, \beta)$ is strictly decreasing in s for each β , the sum of shares s at the optimal allocation is increasing in β . $\square$

B.2 Three-agent networks

In this section, we describe the optimal equity allocation in a three-agent weighted network (without self-links). We begin by describing the optimal allocation and then discuss how this allocation changes as we vary the network.

We first describe which agents receive the most equity and which agents are active as a function of the network structure. We then derive comparative statics of the optimal equity shares as the network structure varies. We will see that the equity given to an agent need not be monotonic with respect to the weights on his links.

Without loss of generality, we can assume the adjacency matrix is

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & G_{13} \\ 1 & 0 & G_{23} \\ G_{13} & G_{23} & 0 \end{bmatrix},$$

with the ordering $1 \geq G_{13} \geq G_{23} \geq 0$.¹ We will consider the case when all three links have positive weight, i.e., $G_{23} > 0$.

Our first result ranks agents' equity allocations and describes the active set:

Proposition 20. *For any $\beta > 0$, there is a unique optimal allocation. The set of active agents in an optimal allocation is $\{1, 2, 3\}$ if and only if*

$$G_{13} + G_{23} > 1.$$

The shares at the optimal allocation are ranked

$$\tau_1^* \geq \tau_2^* \geq \tau_3^*.$$

We defer proofs to the end of this section. The first part of the proposition states the optimal allocation is unique, or equivalently there is a unique best active set for all networks. The second part of the proposition characterizes the active set under the optimal allocation. All agents are active precisely when each agent's degree is greater than the weight of the opposing link, and otherwise only the two higher-degree agents are active. Intuitively, it must be the case that the combination of the two lower-weight links provides complementarities at least as strong as the highest-weight link. Otherwise, the principal prefers to give equity only to the two higher-degree agents.

The final part of the proposition describes which agents receive the most equity. Under any optimal allocation, equity shares are increasing in degree. Intuitively, the weights on links originating from agent 1 dominate those originating from agent 2, which in turn dominate links from agent 3. From the balanced neighborhood equity characterization of the optimal equity incentives, we expect that agents with stronger links receive more equity.²

We now turn to comparative statics of the optimal equity allocation with respect to the

¹To reduce to this case, we can reorder agents and then normalize the link G_{12} to 1.

²Roughly speaking, an agent i 's degree times average equity in i 's neighborhood is equal to a constant. So agents with higher degree need to have neighbors with lower equity. With three agents this means that higher degree implies higher equity.

weight of the weakest link G_{23} , fixing the weight $G_{13} > \frac{1}{2}$. By [Proposition 20](#), agent 3 is active if and only if $G_{23} > 1 - G_{13}$. We will restrict to this interval to take comparative statics, as the optimal allocation does not depend on G_{23} below the threshold $1 - G_{13}$.

Proposition 21. *Fix $G_{13} > \frac{1}{2}$. When $G_{23} \in (1 - G_{13}, G_{13})$, we have*

$$\frac{\partial \left(\frac{\tau_2^*}{\tau_3^*} \right)}{\partial G_{23}} < 0, \quad \text{and} \quad \frac{\partial \left(\frac{\tau_1^*}{\tau_3^*} \right)}{\partial G_{23}} < 0.$$

Moreover, there exists a threshold $g^* \in (0, 1)$, which is a function of G_{13} , such that

$$\frac{\partial \left(\frac{\tau_1^*}{\tau_2^*} \right)}{\partial G_{23}} > 0 \text{ if } G_{23} < g^*(G_{13}) \quad \text{and} \quad \frac{\partial \left(\frac{\tau_1^*}{\tau_2^*} \right)}{\partial G_{23}} < 0 \text{ if } G_{23} > g^*(G_{13}).$$

Under the success probability objective, all equity is allocated and there is no question of how the sum of shares changes; in that case, an immediate implication of [Proposition 21](#) is that the equity of agent 2 is decreasing in G_{23} for small G_{23} , while the equity of agent 1 is decreasing in G_{23} for large G_{23} .

Corollary 4. *In the success probability problem, increasing G_{23} decreases agent 2's optimal share σ_2^* when all agents are active and $G_{23} < g^*(G_{13})$. Increasing G_{23} decreases agent 1's optimal share σ_1^* when $G_{23} > g^*(G_{13})$.*

The first part of the result is more counterintuitive: it says that increasing the strength of an agent's collaborations can decrease the agent's relative equity share. This is what happens to agent 2 when G_{23} is increased starting from a low level. Once G_{23} gets large enough, however, the equity given to agent 2 begins increasing; this is financed by taking equity from agent 1, whose stake is then decreasing in this regime by the second part of the corollary.

B.2.1 Proofs for three-agent networks

We now prove the two propositions in this section.

Proof of Proposition 20. As a preliminary, we prove any optimal active set has at least two agents. Suppose only agent i were active. Let the optimal equity allocation for such an agent be σ^* . The principal then maximizes the output $(1 - \sigma)P(\sigma)$, so σ^* solves $P(\sigma^*) = (1 - \sigma^*)P'(\sigma^*)$ for the residual profits objective or $\sigma^* = 1$ for the success probability objective. But, for any $\beta > 0$, the principal can strictly increase residual profits by giving agents 1 and 2 each shares $\sigma_1 = \sigma_2 = \sigma^*/2$ or increase the success probability by giving shares $\sigma_1 = \sigma_2 = \frac{1}{2}$, which gives a contradiction.

We claim that the unique active set is all three agents if and only if $G_{13} + G_{23} > 1$. If all agents are active at an optimal allocation, then Proposition 2 characterizes the optimal shares as

$$\tau_1^* = \frac{1 + G_{13} - G_{23}}{2G_{13}}c, \quad \tau_2^* = \frac{1 + G_{23} - G_{13}}{2G_{23}}c, \quad \tau_3^* = \frac{G_{23} + G_{13} - 1}{2G_{23}G_{13}}c,$$

for some constant c . A necessary condition for $\{1, 2, 3\}$ to be an optimal active set is $\sigma_3^* > 0$, which is equivalent to $G_{13} + G_{23} > 1$.

To prove the claim, it remains to show this condition is sufficient: if $G_{13} + G_{23} > 1$, then the unique active set is all three agents. Fix an optimal allocation τ^* and let $s = \sum_i \sigma_i^*$ be the corresponding sum of shares. By Lemma 14, this optimal allocation must maximize the constant c in the expression $\tilde{\mathbf{G}}\tau = c\mathbf{1}$ among feasible allocations with $\sum_i \sigma_i = s$.

First suppose the active set is $\{1, 2\}$. By Proposition 2, the two agents receive equal equity shares $\tau_1 = \tau_2 = s/2$ and we obtain $\mathbf{G}\tau = s/2$.

Suppose the principal instead allocated s shares among all three agents so that the balanced neighborhood equity condition is satisfied. This requires

$$\sigma_1 = \frac{1 + G_{13} - G_{23}}{2G_{13}} \cdot c, \sigma_2 = \frac{1 + G_{23} - G_{13}}{2G_{23}} \cdot c, \sigma_3 = \frac{G_{23} + G_{13} - 1}{2G_{23}G_{13}} \cdot c.$$

Since $G_{13} + G_{23} > 1$, the three shares $\sigma_1, \sigma_2, \sigma_3 > 0$, so we have defined a feasible equity allocation. Combining the expressions for the σ_i ,

$$c = \frac{2G_{13}G_{23}s}{2(G_{23} + G_{13}) - 1 - (G_{13} - G_{23})^2} > 0.$$

It remains to check that this allocation gives a higher constant c than the allocation $\sigma_1 = \sigma_2 = s/2$ and $\sigma_3 = 0$ with two active agents. This holds if

$$\begin{aligned} \frac{2G_{13}G_{23}s}{2(G_{23} + G_{13}) - 1 - (G_{13} - G_{23})^2} &> s/2, \\ \iff (G_{13} + G_{23} - 1)^2 &> 0, \end{aligned}$$

and the bottom inequality holds since $G_{13} + G_{23} > 1$. This shows that there is a unique optimal allocation and the active set at this allocation is $\{1, 2, 3\}$.

Finally, we prove that $\tau_1^* \geq \tau_2^* \geq \tau_3^*$ by considering both possible active sets. When all agents are active, by [Proposition 2](#), the optimal shares satisfy

$$\tau_2^* + G_{13}\tau_3^* = \tau_1^* + G_{23}\tau_3^* = G_{13}\tau_1^* + G_{23}\tau_2^*,$$

where $\tau_1^*, \tau_2^*, \tau_3^* > 0$. We have assumed $G_{23} > 0$, so $\tau_1^* \geq \tau_2^* \geq \tau_3^*$.

When only agents 1 and 2 are active, by [Proposition 2](#), the two active agents i and j receive equal shares $\sigma_i^* = \sigma_j^* = \sigma^*$ while agent 3 receives no shares. $\square$

Proof of [Proposition 21](#). Since we are working in the range of values where the optimal active set is all three agents, [Proposition 2](#) implies

$$\frac{\tau_2^*}{\tau_3^*} = \frac{G_{13}(G_{23} + 1 - G_{13})}{G_{23} + G_{13} - 1}.$$

Taking the derivative of the ratio of the optimal shares with respect to G_{23} , we get that

$$\frac{\partial \left(\frac{\tau_2^*}{\tau_3^*} \right)}{\partial G_{23}} = - \frac{2G_{13}(1 - G_{13})}{(G_{23} + G_{13} - 1)^2} < 0.$$

Turning to the relative shares between agent 1 and 3, [Proposition 2](#) implies

$$\frac{\tau_1^*}{\tau_3^*} = \frac{G_{23}(G_{13} + 1 - G_{23})}{G_{23} + G_{13} - 1}.$$

Taking the derivative of this ratio with respect to G_{23} , we get that

$$\frac{\partial \left(\frac{\tau_1^*}{\tau_3^*} \right)}{\partial G_{23}} = \frac{(G_{13} - 1)(1 + G_{13} - G_{23}) - G_{23}(G_{23} + G_{13} - 1)}{(G_{23} + G_{13} - 1)^2} < 0,$$

where the inequality follows from [Proposition 20](#). Finally, we have that

$$\frac{\tau_1^*}{\tau_2^*} = \frac{(1 + G_{13} - G_{23})G_{23}}{(1 + G_{23} - G_{13})G_{13}}.$$

Taking the derivative with respect to G_{23} and analyzing the quadratic function in G_{23} in the numerator gives the result with threshold

$$g^*(G_{13}) = \sqrt{1 - G_{13}} \left(\sqrt{2} - \sqrt{1 - G_{13}} \right). \quad \square$$

B.3 Spectral radius maximization

We now state and prove the connection to the spectral radius maximization problem discussed in [Section 2.5.3](#). Recall the spectral radius of a matrix $\mathbf{M}$, which we write as $\rho(\mathbf{M})$, is the largest magnitude of an eigenvalue of $\mathbf{M}$.

Proposition 22. *If $\boldsymbol{\tau}^*$ is optimal for the success probability objective, then $\mathbf{T}^* = \text{diag}(\boldsymbol{\tau}^*)$ maximizes the spectral radius $\rho(\mathbf{T}\mathbf{G})$ among matrices $\mathbf{T} = \text{diag}(\boldsymbol{\tau})$ such that $\boldsymbol{\tau}$ is a feasible allocation.*

Proof. For the proof, we consider the alternate model (discussed in [Section 2.2](#)) in which the team has a deterministic output $P(Y)$ and the principal allocates equity shares $\sigma_i P(Y)$ to each agent i . We moreover assume that $P(Y) = Y$. Our analysis applies essentially

unchanged, with equilibrium actions under shares τ now given by

$$\mathbf{a}^* = [I - \beta \mathbf{T}\mathbf{G}]^{-1} \tau.$$

In particular, the optimal allocations are unchanged and [Proposition 5](#) continues to apply in this alternate model.

Suppose σ^* is optimal for the output objective for some complementarity parameter β_0 but does not maximize the spectral radius $\rho(\mathbf{T}\mathbf{G})$. Let σ' maximize the spectral radius $\rho(\mathbf{T}\mathbf{G})$. Then, by assumption, $\rho(\Sigma^* \mathbf{G}) < \rho(\Sigma' \mathbf{G})$. Recall that given shares τ , the equilibrium actions are

$$\mathbf{a}^* = [I - \beta \mathbf{T}\mathbf{G}]^{-1} \tau.$$

Since τ is the right eigenvector of $\mathbf{T}\mathbf{G}$ with largest eigenvalue, we have

$$\mathbf{a}^* = \frac{1}{1 - \beta \rho(\mathbf{T}\mathbf{G})} \tau.$$

Taking the limit of β to $1/\rho(\Sigma' \mathbf{G})$ from below, we have $\|\mathbf{a}^*\|_2 \rightarrow \infty$ under the equity allocation τ' but $\|\mathbf{a}^*\|_2$ remains bounded under the equity allocation τ^* . So we can choose β_1 such that the team performance with shares σ' is strictly greater than the team performance with shares σ^* . This implies that σ^* does not maximize the output objective with complementarity parameter β_1 . [Proposition 5](#) implies that σ^* does not maximize the success probability with complementarity parameter β_0 either, giving a contradiction. Thus, any solution to the principal's problem maximizes the spectral radius $\rho(\mathbf{T}\mathbf{G})$. $\square$

Appendix C

Appendix for Network Interventions: Targeting Agents or Targeting Links?

C.1 Omitted Proofs

Proof of Proposition 10. By assumption that the set of subsidy interventions is compact, there exists a $\tilde{t} \geq 0$ such that

$$\begin{aligned} \beta_i &\leq \tilde{t} \quad \text{for all } i \in \mathcal{N}, \text{ and} \\ \sigma_{ij} &\leq \tilde{t} \quad \text{for all } i, j \in \mathcal{N}, \end{aligned}$$

for every subsidy $(\boldsymbol{\beta}, \boldsymbol{\sigma})$ in the set $\mathcal{T}$. Fix a subsidy $(\boldsymbol{\beta}, \boldsymbol{\sigma})$. We will establish the existence of a fixed-point of the agents' best-response correspondence. Given an action profile $\mathbf{a}$ and link intensities $\mathbf{g}$, an agent's utility is strictly concave in link intensity g_{ij} . The unique optimal link intensity is

$$g_{ij}^* = \max \left\{ 0, \frac{(\rho a_i a_j + s_{ij} + \sigma_{ij})}{f_{ij}} \right\}. \quad (\text{C.1})$$

Furthermore, the utility is strictly concave in a_i and the unique optimal action is

$$a_i^* = b_i + \beta_i + \rho \sum_{j \in \mathcal{N} \setminus \{i\}} G_{ij} a_j.$$

Thus, it suffices to establish a fixed point of the best-response map $\Phi : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^n$ defined as

$$(\Phi(\mathbf{a}))_i := b_i + \beta_i + \rho \sum_{j \in \mathcal{N} \setminus \{i\}} G_{ij} a_j \quad \text{for every } i \in \mathcal{N},$$

where $G_{ij} = g_{ij} + g_{ji}$ with g_{ij} defined as in (C.1). Consider the following upper bound on the best-response of any agent:

$$\begin{aligned} (\Phi(\mathbf{a}))_i &\leq \bar{b} + \tilde{t} + \rho \sum_{j \in \mathcal{N} \setminus \{i\}} |G_{ij}| a_j, \\ &\leq \bar{b} + \tilde{t} + \rho \sum_{j \in \mathcal{N} \setminus \{i\}} \frac{2\rho a_i a_j + 2\tilde{t} + 2\bar{s}}{f_{ij}} a_j. \end{aligned}$$

Given curvatures f_{ij} , define $\underline{f} := \min_{i,j} f_{ij}$. Suppose there exists a $R > 0$ such that

$$\bar{b} + \tilde{t} + \frac{2\rho^2}{\underline{f}} n R^3 + \frac{2\rho(\tilde{t} + \bar{s})}{\underline{f}} n R < R. \quad (\text{C.2})$$

Then, applying Brouwer's fixed point theorem on the compact and convex domain $[0, R]^n$, the continuous map $\Phi(\cdot)$ has a fixed point in $[0, R]^n$. When $R = 2(\bar{b} + \tilde{t})$ and

$$\rho \leq \frac{\underline{f}}{16n(\bar{b} + \tilde{t})^2} \cdot \left(\sqrt{\frac{n^2(\tilde{t} + \bar{s})^2}{\underline{f}^2} + \frac{16n(\bar{b} + \tilde{t})^2}{\underline{f}}} - \frac{n(\tilde{t} + \bar{s})}{\underline{f}} \right),$$

equation (C.2) is satisfied. The statement of the result follows. $\square$

Proof of Lemma 3. Consider any subsidy scheme (β, σ) and a corresponding equilibrium

$(\mathbf{a}^*, \mathbf{G}^*)$ that is differentiable in a neighborhood of $(\boldsymbol{\beta}, \boldsymbol{\sigma})$. Applying [Proposition 11](#), for any agent $j \in \mathcal{N}$ whose action incentive constraint binds, the equilibrium action satisfies

$$c_j a_j^* = b_j + \beta_j + \rho \sum_{k \in \mathcal{N} \setminus \{j\}} G_{jk} a_k^*. \quad (\text{C.3})$$

Furthermore, for any individual $k \in \mathcal{N}$ whose link incentive constraint binds with agent $l \in \mathcal{N}$, the equilibrium link intensity satisfies

$$f_{kl} g_{kl}^* = \sigma_{kl} + s_{kl} + \rho a_k^* a_l^*. \quad (\text{C.4})$$

We will implicitly differentiate [\(C.3\)](#) and [\(C.4\)](#) with respect to β_i and σ_{ij} respectively to solve for the derivative of equilibrium actions. We begin with a perturbation in action subsidy β_i . For every agent whose action incentive constraint binds, implicitly differentiating [\(C.3\)](#) with respect to β_i , we have

$$c_j \frac{da_j^*}{d\beta_i} = \rho \left(\sum_{k \in \mathcal{N} \setminus \{j\}} G_{jk}^* \frac{da_k^*}{d\beta_i} + \sum_{k \in \mathcal{N} \setminus \{j\}} \left(\frac{dg_{jk}^*}{d\beta_i} + \frac{dg_{kj}^*}{d\beta_i} \right) a_k^* \right). \quad (\text{C.5})$$

If agent i 's action incentive constraint binds, then

$$c_i \frac{da_i^*}{d\beta_i} = 1 + \rho \left(\sum_{k \in \mathcal{N} \setminus \{i\}} G_{ik}^* \frac{da_k^*}{d\beta_i} + \sum_{k \in \mathcal{N} \setminus \{i\}} \left(\frac{dg_{ik}^*}{d\beta_i} + \frac{dg_{ki}^*}{d\beta_i} \right) a_k^* \right). \quad (\text{C.6})$$

Moreover, implicitly differentiating [\(C.4\)](#) with respect to β_i for every link intensity from agent k to agent l whose incentive constraint binds, we have

$$f_{kl} \frac{dg_{kl}^*}{d\beta_i} = \rho \left(a_k^* \frac{da_l^*}{d\beta_i} + \frac{da_k^*}{d\beta_i} a_l^* \right). \quad (\text{C.7})$$

Substituting [\(C.7\)](#) in [\(C.5\)](#), we have for every $j \in \mathcal{N} \setminus \{i\}$ whose action incentive constraint

binds,

$$c_j \frac{da_j^*}{d\beta_i} = \rho \left(\sum_{k \in \mathcal{N} \setminus \{j\}} G_{jk}^* \frac{da_k^*}{d\beta_i} + \rho \sum_{k \in \mathcal{N} \setminus \{j\}} \frac{2}{f_{jk}} \left(a_j^* \frac{da_k^*}{d\beta_i} + a_k^* \frac{da_j^*}{d\beta_i} \right) a_k^* \right), \quad (\text{C.8})$$

$$= \rho \left(\sum_{k \in \mathcal{N} \setminus \{j\}} \left(G_{jk}^* + \frac{2\rho}{f_{jk}} a_j^* a_k^* \right) \frac{da_k^*}{d\beta_i} + 2\rho \frac{da_j^*}{d\beta_i} \sum_{k \in \mathcal{N} \setminus \{j\}} \frac{(a_k^*)^2}{f_{jk}} \right). \quad (\text{C.9})$$

Analogously, substituting (C.7) in (C.6), we have

$$c_i \frac{da_i^*}{d\beta_i} = 1 + \rho \left(\sum_{k \in \mathcal{N} \setminus \{i\}} \left(G_{ik}^* + \frac{2\rho}{f_{ik}} a_i^* a_k^* \right) \frac{da_k^*}{d\beta_i} + 2\rho \frac{da_i^*}{d\beta_i} \sum_{k \in \mathcal{N} \setminus \{i\}} \frac{(a_k^*)^2}{f_{ik}} \right). \quad (\text{C.10})$$

We combine (C.10) and (C.9) to write the derivative of actions $\mathbf{a}^*$ with respect to β_i in vector form. Thus, when agent i 's action incentive constraint binds,

$$(\mathbf{C} - \rho \mathbf{M}) \frac{d\mathbf{a}^*}{d\beta_i} = \mathbf{e}_i,$$

where $\mathbf{C}$ is a diagonal matrix with $C_{ii} = c_i$, $\mathbf{M}$ is a matrix with entry $M_{ij} = G_{ij}^* + 2\rho a_i^* a_j^* / f_{ij}$ and $M_{ii} = 2\rho \sum_{k \in \mathcal{N} \setminus \{i\}} (a_k^*)^2 / f_{ik}$, and $\mathbf{e}_i$ is the i^{th} basis vector. By assumption, equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ is differentiable at subsidy (β, σ) and thus

$$\frac{d\mathbf{a}^*}{d\beta_i} = (\mathbf{C} - \rho \mathbf{M})_i^{-1},$$

where $(\mathbf{C} - \rho \mathbf{M})_i^{-1}$ is the i^{th} column of matrix $(\mathbf{C} - \rho \mathbf{M})^{-1}$. The welfare effect of perturbing subsidy β_i is

$$\frac{d\mathcal{W}}{d\beta_i} = \nabla \mathcal{W}^\top \frac{d\mathbf{a}^*}{d\beta_i} = \nabla \mathcal{W}^\top (\mathbf{C} - \rho \mathbf{M})_i^{-1}.$$

Call this quantity $\mathcal{R}_i$.

When agent i 's action incentive constraint does not bind, then $d\mathbf{a}^*/d\beta_i = 0$, and consequently, $\frac{d\mathcal{W}}{d\beta_i} = 0$.

We now follow the same set of calculations to solve for the derivative of equilibrium actions $\mathbf{a}^*$ with respect to link subsidy σ_{ij} . For every agent $k \in \mathcal{N}$ whose action incentive constraint binds, implicitly differentiating (C.3) with respect to σ_{ij} , we have

$$c_k \frac{da_k^*}{d\beta_i} = \rho \left(\sum_{l \in \mathcal{N} \setminus \{k\}} G_{kl}^* \frac{da_l^*}{d\sigma_{ij}} + \sum_{l \in \mathcal{N} \setminus \{k\}} \left(\frac{dg_{kl}^*}{d\sigma_{ij}} + \frac{dg_{lk}^*}{d\sigma_{ij}} \right) a_l^* \right), \quad (\text{C.11})$$

We implicitly differentiate (C.4) with respect to σ_{ij} for agents with link intensity constraints that bind. For every link intensity (k, l) that is not (i, j) , we have

$$f_{kl} \frac{dg_{kl}^*}{d\sigma_{ij}} = \rho \left(a_k^* \frac{da_l^*}{d\sigma_{ij}} + \frac{da_k^*}{d\sigma_{ij}} a_l^* \right), \quad (\text{C.12})$$

and for the link intensity from agent i to j , we have

$$f_{kl} \frac{dg_{kl}^*}{d\sigma_{ij}} = 1 + \rho \left(a_k^* \frac{da_l^*}{d\sigma_{ij}} + \frac{da_k^*}{d\sigma_{ij}} a_l^* \right). \quad (\text{C.13})$$

Substituting (C.12) and (C.13) in (C.11), we have

$$c_k \frac{da_k^*}{d\sigma_{ij}} = \rho \left(\sum_{l \in \mathcal{N} \setminus \{k\}} \left(G_{kl}^* + \frac{2\rho}{f_{kl}} a_k^* a_l^* \right) \frac{da_l^*}{d\sigma_{ij}} + 2\rho \frac{da_k^*}{d\sigma_{ij}} \sum_{l \in \mathcal{N} \setminus \{k\}} \frac{(a_l^*)^2}{f_{kl}} \right),$$

for every agent $k \neq i, j$ whose action incentive constraint binds. Furthermore, if agent i 's action incentive constraint binds, then

$$c_i \frac{da_i^*}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} a_j^* + \rho \left(\sum_{l \in \mathcal{N} \setminus \{i\}} \left(G_{il}^* + \frac{2\rho}{f_{il}} a_i^* a_l^* \right) \frac{da_l^*}{d\sigma_{ij}} + 2\rho \frac{da_i^*}{d\sigma_{ij}} \sum_{l \in \mathcal{N} \setminus \{i\}} \frac{(a_l^*)^2}{f_{il}} \right),$$

and if agent j 's action incentive constraint binds,

$$c_j \frac{da_j^*}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} a_i^* + \rho \left(\sum_{l \in \mathcal{N} \setminus \{j\}} \left(G_{jl}^* + \frac{2\rho}{f_{jl}} a_j^* a_l^* \right) \frac{da_l^*}{d\sigma_{ij}} + 2\rho \frac{da_j^*}{d\sigma_{ij}} \sum_{l \in \mathcal{N} \setminus \{j\}} \frac{(a_l^*)^2}{f_{jl}} \right).$$

Suppose the link intensity incentive constraint from agent i to agent j binds. Furthermore, suppose both agent i and j 's action incentive constraints bind. We combine the set of equations above to write the derivative of actions $\mathbf{a}^*$ with respect to σ_{ij} in vector form. Thus,

$$(\mathbf{C} - \rho \mathbf{M}) \frac{d\mathbf{a}^*}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} (a_j^* \mathbf{e}_i + a_i^* \mathbf{e}_j).$$

By assumption, equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ is differentiable at subsidy $(\boldsymbol{\beta}, \boldsymbol{\sigma})$ and thus

$$\frac{d\mathbf{a}^*}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} \left(a_j^* (\mathbf{C} - \rho \mathbf{M})_i^{-1} + a_i^* (\mathbf{C} - \rho \mathbf{M})_j^{-1} \right).$$

Using this expression, the welfare effect of perturbing subsidy σ_{ij} is

$$\frac{d\mathcal{W}}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} (\mathcal{R}_i a_j^* + \mathcal{R}_j a_i^*).$$

The statement of the result follows in this case.

Next, suppose the link intensity incentive constraint from agent i to j binds and agent i 's action incentive constraint binds but agent j 's action incentive does not, i.e., $a_j^* = 0$. Thus,

$$\frac{d\mathbf{a}^*}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} a_j^* (\mathbf{C} - \rho \mathbf{M})_i^{-1} = \mathbf{0}$$

and consequently $d\mathcal{W}/d\sigma_{ij} = 0$. Since, $d\mathbf{a}^*/d\beta_j = \mathbf{0}$, the statement of the result holds in this case as well. We get the same conclusion when agent j 's action incentive constraint binds but agent i 's action incentive does not.

Next, if neither of agent i and j 's action incentive constraints bind, then $d\mathbf{a}^*/d\beta_i = d\mathbf{a}^*/d\beta_j = d\mathbf{a}^*/d\sigma_{ij} = \mathbf{0}$. An identical equation is satisfied when the link intensity constraint from agent i to j does not bind. The statement of the result follows. $\square$

Proof of Theorem 4. For a given welfare objective $\mathcal{W}(\cdot)$, the Lagrangian of the planner's

optimization problem is given by

$$\mathcal{L}(\mathbf{a}, \mathbf{g}, \boldsymbol{\beta}, \boldsymbol{\sigma}, \lambda) = \mathcal{W}(\mathbf{a}) - \lambda \left(\sum_{k \in \mathcal{N}} \beta_k a_k + \sum_{k \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{k\}} \sigma_{kl} G_{kl} \right).$$

We begin by taking the derivative of the Lagrangian with respect to β_i :

$$\frac{d\mathcal{L}}{d\beta_i} = \nabla \mathcal{W}^\top \frac{d\mathbf{a}^*}{d\beta_i} - \lambda \left(\sum_{k \in \mathcal{N}} \beta_k \frac{da_k^*}{d\beta_i} + \sum_{k \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{k\}} \sigma_{kl} \frac{dG_{kl}^*}{d\beta_i} \right) - \lambda a_i^*. \quad (\text{C.14})$$

Substituting (C.7) in the expression above, there is a vector $\boldsymbol{\omega}$ such that the derivative of the lagrangian with respect to β_i satisfies

$$\frac{d\mathcal{L}}{d\beta_i} = \boldsymbol{\omega}^\top \frac{d\mathbf{a}^*}{d\beta_i} - \lambda a_i^*. \quad (\text{C.15})$$

Furthermore, when agent i 's action incentive constraint binds, substituting the expression

$$\frac{d\mathbf{a}^*}{d\beta_i} = (\mathbf{C} - \rho \mathbf{M})_i^{-1},$$

derived in the proof of Lemma 3, into (C.15) we have

$$\frac{d\mathcal{L}}{d\beta_i} = \boldsymbol{\omega}^\top (\mathbf{C} - \rho \mathbf{M})_i^{-1} - \lambda a_i^*, \quad (\text{C.16})$$

$$= \mathcal{R}'_i - \lambda a_i^*, \quad (\text{C.17})$$

where we define $\mathcal{R}'_i := \boldsymbol{\omega}^\top (\mathbf{C} - \rho \mathbf{M})_i^{-1}$.

We next take the derivative of the Lagrangian with respect to σ_{ij} :

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \nabla \mathcal{W}^\top \frac{d\mathbf{a}^*}{d\sigma_{ij}} - \lambda \left(\sum_{k \in \mathcal{N}} \beta_k \frac{da_k^*}{d\sigma_{ij}} + \sum_{k \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{k\}} \sigma_{kl} \frac{dG_{kl}^*}{d\sigma_{ij}} \right) - \lambda G_{ij}^*. \quad (\text{C.18})$$

When $s_{ij} \geq 0$, the link intensity incentive constraint between agents binds. Furthermore, when $s_{ij} < 0$, we assume that $g_{ij}^* > 0$ and $g_{ji}^* > 0$. Thus, for the analysis below, suppose the

link intensity constraint from agent i to j and from agent j to i binds.

We substitute (C.12) and (C.13) into (C.18) and use

$$G_{ij}^* = \frac{(2s_{ij} + \sigma_{ij} + \sigma_{ji} + 2\rho a_i^* a_j^*)}{f_{ij}}.$$

Consider the following chain of equations:

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \boldsymbol{\omega}^\top \frac{d\mathbf{a}^*}{d\sigma_{ij}} - \lambda \frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}} - \lambda G_{ij}^*, \quad (\text{C.19})$$

$$= \boldsymbol{\omega}^\top \frac{d\mathbf{a}^*}{d\sigma_{ij}} - 2\lambda \frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}} - 2\lambda \frac{(s_{ij} + \rho a_i^* a_j^*)}{f_{ij}}, \quad (\text{C.20})$$

$$= \boldsymbol{\omega}^\top \frac{d\mathbf{a}^*}{d\sigma_{ij}} - \frac{2\lambda\rho}{f_{ij}} a_i^* a_j^* - \frac{2\lambda}{f_{ij}} s_{ij} - 2\lambda \frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}}. \quad (\text{C.21})$$

Consider any optimal subsidy scheme $(\boldsymbol{\beta}^*, \boldsymbol{\sigma}^*)$ and corresponding equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ such that the budget constraint binds, that is, $\lambda > 0$. Suppose that the action incentive constraints for both agents i and j bind. Then

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} (\mathcal{R}'_i a_j^* + \mathcal{R}'_j a_i^* - 2\lambda a_i^* a_j^*) - \frac{2\lambda}{f_{ij}} s_{ij} - 2\lambda \frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}}. \quad (\text{C.22})$$

Furthermore, analyzing (C.17), the KKT first-order conditions imply

$$\mathcal{R}'_i \leq \lambda a_i^* \quad \text{and} \quad \mathcal{R}'_j \leq \lambda a_j^*.$$

Substituting in (C.22), we have

$$\frac{d\mathcal{L}}{d\sigma_{ij}} \leq -\frac{2\lambda}{f_{ij}} s_{ij} - 2\lambda \frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}}.$$

An analogous calculation shows that the derivative of the lagrangian with respect to σ_{ji} satisfies

$$\frac{d\mathcal{L}}{d\sigma_{ji}} \leq -\frac{2\lambda}{f_{ij}} s_{ij} - 2\lambda \frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}}.$$

Thus, if $s_{ij} \geq 0$, it must be that $\sigma_{ij}^* = \sigma_{ji}^* = 0$. Otherwise, the total subsidy across the pair of agents $\sigma_{ij}^* + \sigma_{ji}^* > 0$, implying that the Lagrangian has a negative derivative with respect to the positive link subsidy, a contradiction.

In contrast, suppose $s_{ij} < 0$ and $\beta_i^* > 0$ and $\beta_j^* > 0$. By the KKT first-order conditions, $\mathcal{R}'_i = \lambda a_i^*$ and $\mathcal{R}'_j = \lambda a_j^*$. Substituting in (C.21), we have

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = -\frac{2\lambda}{f_{ij}}s_{ij} - 2\lambda\frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}}.$$

If $\sigma_{ij}^* + \sigma_{ji}^* = 0$, then $d\mathcal{L}/d\sigma_{ij} > 0$ which contradicts the KKT first-order conditions. Thus, it must be that the total link subsidy $\sigma_{ij}^* + \sigma_{ji}^* > 0$. The statement of the result follows.

Next, suppose that the action incentive constraint for only agent i binds. (The case when the action incentive constraint for only agent j binds follows symmetrically.) Thus, we have $a_j^* = 0$. Furthermore, as established in the proof of Lemma 3, $d\mathbf{a}^*/d\sigma_{ij} = \mathbf{0}$. Substituting in (C.21), we have

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = -\frac{2\lambda}{f_{ij}}s_{ij} - \frac{2\lambda}{f_{ij}}(\sigma_{ij} + \sigma_{ji}).$$

An argument identical to one in the paragraph above establishes that if $s_{ij} \geq 0$, then $\sigma_{ij}^* = \sigma_{ji}^* = 0$. We now show why Part (ii) of the result holds. Since $a_j^* = 0$ and by assumption that the link intensity incentive constraint binds, the link intensity $g_{ij}^* = (s_{ij} + \sigma_{ij})/f_{ij}$ and $g_{ji}^* = (s_{ij} + \sigma_{ji})/f_{ij}$. In the case $s_{ij} < 0$, because $g_{ij}^*, g_{ji}^* > 0$, it must be $\sigma_{ij}^* > 0$ and $\sigma_{ji}^* > 0$.

The case where neither agent i 's nor agent j 's action incentive constraint binds follows analogously, completing the proof $\square$

Proof of Theorem 5. By assumption, throughout this proof, the link incentive constraint between i and j binds. Furthermore, since the baseline incentive to take an action b_i is non-negative, the action incentive constraint for agents' binds. We first characterize an equilibrium strategy $(\mathbf{a}^*, \mathbf{g}^*)$ for a given subsidy profile $(\boldsymbol{\beta}, \boldsymbol{\sigma})$. For any link intensity g_{kl}

such that the equilibrium first-order condition binds, we have

$$f_{kl}g_{kl}^* = s_{kl} + \sigma_{kl}.$$

Furthermore, for any action a_k^* , we have

$$a_k^* = b_k + \beta_k + \rho \sum_{l \in \mathcal{N} \setminus \{k\}} G_{kl}^* a_l^*.$$

Rewriting the above system of equations, equilibrium actions $\mathbf{a}^*$ must solve

$$(\mathbf{I} - \rho \mathbf{G}^*) \mathbf{a}^* = \mathbf{b} + \boldsymbol{\beta} \tag{C.23}$$

where $\mathbf{I}$ is the identity matrix. By assumption, equilibrium actions are differentiable. Thus, by the implicit function theorem, the matrix $(\mathbf{I} - \rho \mathbf{G}^*)$ is invertible. Furthermore, we will establish that the spectral radius of $\mathbf{G}^*$, which we denote by $\mu_1(\mathbf{G}^*)$, is at most one. The network $\mathbf{G}^*$ is non-negative. Thus, by the Perron-Frobenius theorem, the spectral radius $\mu_1(\mathbf{G}^*)$ is an eigenvalue of $\mathbf{G}^*$ with a corresponding eigenvector $\mathbf{v}_1$ having all entries non-negative. Multiplying (C.23) by $\mathbf{v}_1^\top$, we have

$$(1 - \rho \mu_1(\mathbf{G}^*)) \mathbf{v}_1^\top \mathbf{a}^* = \mathbf{v}_1^\top (\mathbf{b} + \boldsymbol{\beta}).$$

In the equation above, because $b_i + \beta_i \geq \underline{b}$ for every $i \in \mathcal{N}$ and $\mathbf{v}_1$ has unit l_2 -norm, the right-hand side is strictly positive. Thus, we have $\mu_1(\mathbf{G}^*) < 1/\rho$.

Analogous to the endogenous setting, we employ a first-order approach to solve for an optimal subsidy scheme. We begin by analyzing the derivative of equilibrium actions $\mathbf{a}^*$ with respect to β_i . A straightforward calculation establishes that

$$\frac{d\mathbf{a}^*}{d\beta_i} = (\mathbf{I} - \rho \mathbf{G}^*)_i^{-1},$$

where $(\mathbf{I} - \rho \mathbf{G}^*)_i^{-1}$ is the i^{th} column of the matrix $(\mathbf{I} - \rho \mathbf{G}^*)^{-1}$.

A similar calculation establishes that the derivative of equilibrium actions with respect to

σ_{ij} satisfies

$$\frac{d\mathbf{a}^*}{d\sigma_{ij}} = \frac{\rho}{f_{ij}} \left((\mathbf{I} - \rho\mathbf{G}^*)_i^{-1} a_j^* + (\mathbf{I} - \rho\mathbf{G}^*)_j^{-1} a_i^* \right).$$

For a given welfare objective $\mathcal{W}(\cdot)$, the Lagrangian of the planner's optimization problem is given by

$$\mathcal{L}(\mathbf{a}, \mathbf{g}, \boldsymbol{\beta}, \boldsymbol{\sigma}, \lambda) = \mathcal{W}(\mathbf{a}) - \lambda \left(\sum_{k \in \mathcal{N}} \beta_k a_k + \sum_{k \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{k\}} \sigma_{kl} G_{kl} \right).$$

We begin by taking the derivative of the Lagrangian with respect to β_i :

$$\frac{d\mathcal{L}}{d\beta_i} = \nabla \mathcal{W}^\top \frac{d\mathbf{a}^*}{d\beta_i} - \lambda \sum_{k \in \mathcal{N}} \beta_k \frac{da_k^*}{d\beta_i} - \lambda a_i^*, \quad (\text{C.24})$$

$$= \mathcal{R}'_i - \lambda a_i^*, \quad (\text{C.25})$$

where $\mathcal{R}'_i := (\nabla \mathcal{W} - \lambda \boldsymbol{\beta})^\top (\mathbf{I} - \rho\mathbf{G}^*)_i^{-1}$. Next, we take the derivative of the Lagrangian with respect to σ_{ij} :

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \nabla \mathcal{W}^\top \frac{d\mathbf{a}^*}{d\sigma_{ij}} - \lambda \sum_{k \in \mathcal{N}} \beta_k \frac{da_k^*}{d\sigma_{ij}} - \lambda G_{ij}^* - \lambda \frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}}, \quad (\text{C.26})$$

$$= \frac{\rho}{f_{ij}} (\mathcal{R}'_i a_j^* + \mathcal{R}'_j a_i^*) - 2\lambda \frac{s_{ij}}{f_{ij}} - 2\lambda \frac{(\sigma_{ij} + \sigma_{ji})}{f_{ij}}. \quad (\text{C.27})$$

Consider any optimal subsidy scheme $(\boldsymbol{\beta}^*, \boldsymbol{\sigma}^*)$ and corresponding equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ such that the budget constraint binds, that is, $\lambda > 0$.

Consider any sequence of environments $\{\mathcal{E}_m\}_{m=1}^\infty$ and a corresponding sequence of optimal interventions $\{(\boldsymbol{\beta}_m^*, \boldsymbol{\sigma}_m^*)\}_{m=1}^\infty$. Let $\mathbf{G}_m^*$ denote the network at the optimal intervention. By assumption, the limit of the spectral radius of $\rho\mathbf{G}_m^*$ converges to one, that is,

$$\rho \cdot \lim_{m \rightarrow \infty} \mu_1(\mathbf{G}_m^*) = 1.$$

For the rest of the proof, we will omit the index of the sequence for notational convenience.

Suppose $\beta_i^* > 0$ and $\beta_j^* > 0$. The KKT first-order conditions imply that $\mathcal{R}'_i = \lambda a_i^*$ and $\mathcal{R}'_j = \lambda a_j^*$. Substituting in (C.27), the derivative of the Lagrangian with respect to σ_{ij} is

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \frac{2\lambda}{f_{ij}} (\rho a_i^* a_j^* - s_{ij} - (\sigma_{ij} + \sigma_{ji})).$$

We will now show that as $m \rightarrow \infty$, every action $a_i^* \rightarrow \infty$. To that end, we will analyze the the following entrywise lower bound on equilibrium acitons:

$$\mathbf{a}^* = (\mathbf{I} - \rho \mathbf{G}^*)^{-1} (\mathbf{b} + \boldsymbol{\beta}) \quad (\text{from (C.23)}), \quad (\text{C.28})$$

$$= \sum_{k=0}^{\infty} (\rho \mathbf{G}^*)^k (\mathbf{b} + \boldsymbol{\beta}) \quad (\text{using } \rho \mu(\mathbf{G}^*) < 1) \quad (\text{C.29})$$

$$\geq \underline{b} (\mathbf{I} - \rho \mathbf{G}^*)^{-1} \mathbf{1} \quad (\text{using } b_i \geq \underline{b}). \quad (\text{C.30})$$

First, using the property of uniform primitivity on the sequence of matrices, we will establish a uniform lowerbound on the entry of the eigenvector $\mathbf{v}_1$ corresponding to the largest eigenvalue $\rho \mu_1(\mathbf{G}^*)$. By uniform primitivity, there exists a constant $L \in \mathbb{N}$ and a $\delta > 0$ such that

$$(\mathbf{G}^*)^L \geq \delta \mu_1(\mathbf{G}^*)^L \mathbf{1} \mathbf{1}^\top \quad (\text{C.31})$$

entrywise for every matrix in the sequence. Consider the following chain of inequalities:

$$\begin{aligned} (\rho \mu_1(\mathbf{G}^*))^L \mathbf{v}_1 &= (\rho \mathbf{G}^*)^L \mathbf{v}_1, \\ &\geq \delta (\rho \mu_1(\mathbf{G}^*))^L \left(\sum_{i \in \mathcal{N}} v_{1i} \right) \mathbf{1} \quad \text{from (C.31)}. \end{aligned}$$

Thus, we have $\mathbf{v}_1 \geq \delta \left(\sum_{i \in \mathcal{N}} v_{1i} \right) \mathbf{1}$ entrywise for matrices in the sequence with a large enough index so that $\rho \mu_1(\mathbf{G}^*)$ is arbitrarily close to 1. Since $\mathbf{v}_1$ has unit l_2 -norm and is entrywise non-negative, the sum of entries $\sum_{i \in \mathcal{N}} v_{1i} \geq 1$. Consequently, $\mathbf{v}_1 \geq \delta \mathbf{1}$ entrywise.

Next, we will use the spectral decomposition of $\mathbf{G}^*$ to solve for the vector $(\mathbf{I} - \rho \mathbf{G}^*)^{-1} \mathbf{1}$ in

(C.30). Consider the following chain of inequalities:

$$\begin{aligned}
 (\mathbf{I} - \rho \mathbf{G}^*)^{-1} \mathbf{1} &= \frac{\mathbf{v}_1 \mathbf{v}_1^\top \mathbf{1}}{1 - \rho \mu_1(\mathbf{G}^*)} + \sum_{k=2}^n \frac{\mathbf{v}_k \mathbf{v}_k^\top \mathbf{1}}{1 - \rho \mu_k(\mathbf{G}^*)}, \\
 &\geq \underbrace{\frac{\delta}{1 - \rho \mu_1(\mathbf{G}^*)} \mathbf{1}}_{\mathcal{P}} + \underbrace{\sum_{k=2}^n \frac{\mathbf{v}_k \mathbf{v}_k^\top \mathbf{1}}{1 - \rho \mu_k(\mathbf{G}^*)}}_{\mathcal{R}} \quad (\text{using } \mathbf{v}_1 \geq \delta \mathbf{1} \text{ and } \mathbf{v}_1^\top \mathbf{1} \geq 1).
 \end{aligned}$$

By assumption (c) that the limit of the spectral gap is positive, every entry in the vector $\mathcal{R}$ is bounded. By assumption (b) that $\lim_{m \rightarrow \infty} \rho \mu_1(\mathbf{G}^*) = 1$, the vector $\mathcal{P}$ entrywise converges to ∞ . Substituting in (C.30), for every $s > 0$ there exists a $m^* \in \mathbb{N}$ such that for all $m > m^*$, every pair of agents i and j have $a_i^* a_j^* > s$.

We will now analyze the derivative of the Lagrangian with respect to σ_{ij} . Recall,

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \frac{2\lambda}{f_{ij}} (\rho a_i^* a_j^* - s_{ij} - (\sigma_{ij} + \sigma_{ji})). \quad (\text{C.32})$$

Suppose $\sigma_{ij}^* = \sigma_{ji}^* = 0$. Substituting in (C.32), we have

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \frac{2\lambda}{f_{ij}} (\rho a_i^* a_j^* - s_{ij}).$$

Since $s_{ij} \leq \bar{s}$, the derivative of the Lagrangian $d\mathcal{L}/d\sigma_{ij} > 0$ for m large enough, as established in the previous paragraph. But this violates the KKT first-order conditions. Thus, it must be that $\sigma_{ij}^* + \sigma_{ji}^* > 0$. $\square$

Proof of Theorem 6. The proof follows an identical approach to that of proving Theorem 4. We consider the case when the incentive constraints bind for i and j 's action, and for link intensities between i and j . The remaining cases are identical to the proof of Theorem 4.

For every agent $j \in \mathcal{N}$, we have

$$C'_j(a_j^*) = b_j + \beta_j + \rho \sum_{k \in \mathcal{N} \setminus \{j\}} G_{jk}^* u'_{jk}(a_j^*) \tilde{u}_{jk}(a_k^*) \quad (\text{C.33})$$

Furthermore,

$$F'_{kl}(g_{kl}^*) = \sigma_{kl} + s_{kl} + \rho u_{kl}(a_k^*) \tilde{u}_{kl}(a_l^*) \quad (\text{C.34})$$

for every pair of individuals $k \in \mathcal{N}$ and $l \in \mathcal{N}$. We will implicitly differentiate (C.33) and (C.34) with respect to β_i and σ_{ij} to solve for the derivative of equilibrium actions. We begin by differentiating with respect to the action subsidy β_i . For every agent $j \in \mathcal{N} \setminus \{i\}$, implicitly differentiating (C.33) with respect to β_i , we have

$$\begin{aligned} C_j''(a_j^*) \frac{da_j^*}{d\beta_i} = \rho \sum_{k \in \mathcal{N} \setminus \{j\}} G_{jk}^* \left(u_{jk}''(a_j^*) \tilde{u}_{jk}(a_k^*) \frac{da_j^*}{d\beta_i} + u_{jk}'(a_j^*) \tilde{u}_{jk}'(a_k^*) \frac{da_k^*}{d\beta_i} \right) \\ + \rho \sum_{k \in \mathcal{N} \setminus \{j\}} \frac{dG_{jk}^*}{d\beta_i} u_{jk}'(a_j^*) \tilde{u}_{jk}(a_k^*), \quad (\text{C.35}) \end{aligned}$$

and

$$\begin{aligned} C_i''(a_i^*) \frac{da_i^*}{d\beta_i} = 1 + \rho \sum_{k \in \mathcal{N} \setminus \{i\}} G_{ik}^* \left(u_{ik}''(a_i^*) \tilde{u}_{ik}(a_k^*) \frac{da_i^*}{d\beta_i} + u_{ik}'(a_i^*) \tilde{u}_{ik}'(a_k^*) \frac{da_k^*}{d\beta_i} \right) \\ + \rho \sum_{k \in \mathcal{N} \setminus \{i\}} \frac{dG_{ik}^*}{d\beta_i} u_{ik}'(a_i^*) \tilde{u}_{ik}(a_k^*). \quad (\text{C.36}) \end{aligned}$$

Moreover, implicitly differentiating (C.34) with respect to β_i , for every link intensity, we have

$$F_{kl}''(g_{kl}^*) \frac{dg_{kl}^*}{d\beta_i} = \rho \left(u_{kl}'(a_k^*) \tilde{u}_{kl}(a_l^*) \frac{da_k^*}{d\beta_i} + u_{kl}(a_k^*) \tilde{u}_{kl}'(a_l^*) \frac{da_l^*}{d\beta_i} \right). \quad (\text{C.37})$$

Substitute (C.37) in (C.35) and (C.36) to solve the system of equations in variables $\left(\frac{da_k^*}{d\beta_i} \right)_{k \in \mathcal{N}}$. Consequently, there is a matrix $\mathbf{M}$ such that

$$(\mathbf{C} - \rho \mathbf{M}) \frac{d\mathbf{a}^*}{d\beta_i} = \mathbf{e}_i,$$

where $\mathbf{C}$ is a diagonal matrix with $C_{ii} = C_i''(a_i^*)$, and $\mathbf{e}_i$ is the i^{th} basis vector. By assumption,

equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ is differentiable at subsidy $(\boldsymbol{\beta}, \boldsymbol{\sigma})$ and thus

$$\frac{d\mathbf{a}^*}{d\beta_i} = (\mathbf{C} - \rho\mathbf{M})_i^{-1},$$

where $(\mathbf{C} - \rho\mathbf{M})_i^{-1}$ is the i^{th} column of matrix $(\mathbf{C} - \rho\mathbf{M})^{-1}$.

We now follow the same set of calculations to solve for the derivative of equilibrium actions $\mathbf{a}^*$ with respect to link subsidy σ_{ij} . For every agent $k \in \mathcal{N}$, implicitly differentiating (C.33) with respect to σ_{ij} , we have

$$\begin{aligned} C_k''(a_k^*) \frac{da_k^*}{d\sigma_{ij}} = \rho \sum_{l \in \mathcal{N} \setminus \{k\}} G_{kl}^* \left(u_{kl}''(a_k^*) \tilde{u}_{kl}(a_l^*) \frac{da_k^*}{d\sigma_{ij}} + u_{kl}'(a_k^*) \tilde{u}_{kl}'(a_l^*) \frac{da_l^*}{d\sigma_{ij}} \right) \\ + \rho \sum_{l \in \mathcal{N} \setminus \{k\}} \frac{dG_{kl}^*}{d\sigma_{ij}} u_{kl}'(a_k^*) \tilde{u}_{kl}(a_l^*). \end{aligned} \quad (\text{C.38})$$

Moreover, implicitly differentiating (C.34) with respect to σ_{ij} , for every link intensity from agent k to l , excluding that from agent i to j , we have

$$F_{kl}''(g_{kl}^*) \frac{dg_{kl}^*}{d\sigma_{ij}} = \rho \left(u_{kl}'(a_k^*) \tilde{u}_{kl}(a_l^*) \frac{da_k^*}{d\sigma_{ij}} + u_{kl}(a_k^*) \tilde{u}_{kl}'(a_l^*) \frac{da_l^*}{d\sigma_{ij}} \right), \quad (\text{C.39})$$

and for the link intensity from agent i to j , we have

$$F_{ij}''(g_{ij}^*) \frac{dg_{ij}^*}{d\sigma_{ij}} = 1 + \rho \left(u_{ij}'(a_i^*) \tilde{u}_{ij}(a_j^*) \frac{da_i^*}{d\sigma_{ij}} + u_{ij}(a_i^*) \tilde{u}_{ij}'(a_j^*) \frac{da_j^*}{d\sigma_{ij}} \right). \quad (\text{C.40})$$

Substitute (C.40) and (C.39) in (C.38) to solve the system of equations. Consequently, using the assumption that $u_{ij}(\cdot) = \tilde{u}_{ji}(\cdot)$ we have

$$(\mathbf{C} - \rho\mathbf{M}) \frac{d\mathbf{a}^*}{d\sigma_{ij}} = \rho \frac{1}{F_{ij}''(g_{ij}^*)} (u_{ij}'(a_i^*) \tilde{u}_{ij}(a_j^*) \mathbf{e}_i + u_{ij}(a_i^*) \tilde{u}_{ij}'(a_j^*) \mathbf{e}_j),$$

where $\mathbf{e}_i$ and $\mathbf{e}_j$ are standard basis vectors. By assumption, equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ is differen-

tiable at subsidy (β, σ) and thus

$$\frac{d\mathbf{a}^*}{d\sigma_{ij}} = \rho \underbrace{\frac{1}{F''_{ij}(g_{ij}^*)}}_{\tilde{f}_{ij}} \left(u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) (\mathbf{C} - \rho \mathbf{M})_i^{-1} + u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) (\mathbf{C} - \rho \mathbf{M})_j^{-1} \right). \quad (\text{C.41})$$

For a given welfare objective $\mathcal{W}(\cdot)$, the Lagrangian of the planner's optimization problem is given by

$$\mathcal{L}(\mathbf{a}, \mathbf{g}, \beta, \sigma, \lambda) = \mathcal{W}(\mathbf{a}) - \lambda \left(\sum_{k \in \mathcal{N}} \beta_k a_k + \sum_{k \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{k\}} \sigma_{kl} G_{kl} \right).$$

We begin by taking the derivative of the Lagrangian with respect to β_i :

$$\frac{d\mathcal{L}}{d\beta_i} = \nabla \mathcal{W}^\top \frac{d\mathbf{a}^*}{d\beta_i} - \lambda \left(\sum_{k \in \mathcal{N}} \beta_k \frac{da_k^*}{d\beta_i} + \sum_{k \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{k\}} \sigma_{kl} \frac{dG_{kl}^*}{d\beta_i} \right) - \lambda a_i^*. \quad (\text{C.42})$$

Substituting (C.37) in the expression above, there is a vector $\boldsymbol{\omega}$ such that the derivative of the lagrangian with respect to β_i satisfies

$$\frac{d\mathcal{L}}{d\beta_i} = \boldsymbol{\omega}^\top \frac{d\mathbf{a}^*}{d\beta_i} - \lambda a_i^*. \quad (\text{C.43})$$

Furthermore, substituting the expression

$$\frac{d\mathbf{a}^*}{d\beta_i} = (\mathbf{C} - \rho \mathbf{M})_i^{-1},$$

into (C.43) we have

$$\frac{d\mathcal{L}}{d\beta_i} = \boldsymbol{\omega}^\top (\mathbf{C} - \rho \mathbf{M})_i^{-1} - \lambda a_i^*, \quad (\text{C.44})$$

$$= \mathcal{R}'_i - \lambda a_i^*, \quad (\text{C.45})$$

where we define $\mathcal{R}'_i := \boldsymbol{\omega}^\top (\mathbf{C} - \rho \mathbf{M})_i^{-1}$.

We next take the derivative of the Lagrangian with respect to σ_{ij} :

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \nabla \mathcal{W}^\top \frac{d\mathbf{a}^*}{d\sigma_{ij}} - \lambda \left(\sum_{k \in \mathcal{N}} \beta_k \frac{da_k^*}{d\sigma_{ij}} + \sum_{k \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{k\}} \sigma_{kl} \frac{dG_{kl}^*}{d\sigma_{ij}} \right) - \lambda G_{ij}^*. \quad (\text{C.46})$$

Substituting (C.41) into (C.46), we have

$$\frac{d\mathcal{L}}{d\sigma_{ij}} = \boldsymbol{\omega}^\top \frac{d\mathbf{a}^*}{d\sigma_{ij}} - \lambda \tilde{f}_{ij} (\sigma_{ij} + \sigma_{ji}) - \lambda G_{ij}^*, \quad (\text{C.47})$$

$$= \rho \tilde{f}_{ij} (\mathcal{R}'_i u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \mathcal{R}'_j u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*)) - \lambda \tilde{f}_{ij} (\sigma_{ij} + \sigma_{ji}) - \lambda G_{ij}^*. \quad (\text{C.48})$$

An analogous calculation will establish that

$$\frac{d\mathcal{L}}{d\sigma_{ji}} = \rho \tilde{f}_{ji} (\mathcal{R}'_j u'_{ji}(a_j^*) \tilde{u}_{ji}(a_i^*) + \mathcal{R}'_i u_{ji}(a_j^*) \tilde{u}'_{ji}(a_i^*)) - \lambda \tilde{f}_{ji} (\sigma_{ij} + \sigma_{ji}) - \lambda G_{ij}^*, \quad (\text{C.49})$$

$$= \rho \tilde{f}_{ji} (\mathcal{R}'_i u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \mathcal{R}'_j u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*)) - \lambda \tilde{f}_{ji} (\sigma_{ij} + \sigma_{ji}) - \lambda G_{ij}^*, \quad (\text{C.50})$$

where the second equality follows from the symmetry assumption that $u_{ij}(\cdot) = \tilde{u}_{ji}(\cdot)$.

We will now establish optimal subsidies by solving for the KKT first-order conditions. Instead of analyzing the derivative of the objective with respect to each individual link intensity σ_{ij} , we analyze the sum of the derivatives across pairs of agents. Adding (C.50) and (C.48), we get

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \sigma_{ij}} + \frac{\partial \mathcal{L}}{\partial \sigma_{ji}}}_{\Delta_{ij} \mathcal{L}} = \rho (\tilde{f}_{ij} + \tilde{f}_{ji}) (\mathcal{R}'_i u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \mathcal{R}'_j u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) - \lambda (\sigma_{ij} + \sigma_{ji})) - 2\lambda G_{ij}^*. \quad (\text{C.51})$$

Consider any optimal subsidy $(\boldsymbol{\beta}^*, \boldsymbol{\sigma}^*)$ and corresponding equilibrium $(\mathbf{a}^*, \mathbf{g}^*)$ such that the budget constraint binds, that is, $\lambda > 0$. Analyzing (C.45), the KKT first-order conditions

imply

$$\mathcal{R}'_i \leq \lambda a_i^* \quad \text{and} \quad \mathcal{R}'_j \leq \lambda a_j^*$$

Substituting in (C.51), we have

$$\Delta_{ij}\mathcal{L} \leq \lambda \left(\tilde{f}_{ij} + \tilde{f}_{ji} \right) \left(\rho a_i^* u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \rho a_j^* u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) - \frac{2G_{ij}^*}{\tilde{f}_{ij} + \tilde{f}_{ji}} - \sigma_{ij} - \sigma_{ji} \right).$$

Suppose the cost $F_{ij}(\cdot)$ is super-quadratic. Then, this equation can be bounded and rearranged as

$$\begin{aligned} \frac{\Delta_{ij}\mathcal{L}}{\lambda \left(\tilde{f}_{ij} + \tilde{f}_{ji} \right)} &\leq \rho a_i^* u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \rho a_j^* u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) - 2F'_{ij}(g_{ij}^*) \frac{F''_{ij}(g_{ji}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} \\ &\quad - 2F'_{ij}(g_{ji}^*) \frac{F''_{ij}(g_{ij}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} - \sigma_{ij} - \sigma_{ji}, \end{aligned}$$

where we substituted $\tilde{f}_{ij} + \tilde{f}_{ji} = 1/F''_{ij}(g_{ij}^*) + 1/F''_{ij}(g_{ji}^*)$. Substituting the expression of equilibrium link intensities in (C.34), we have

$$\begin{aligned} \frac{\Delta_{ij}\mathcal{L}}{\lambda \left(\tilde{f}_{ij} + \tilde{f}_{ji} \right)} &\leq \rho a_i^* u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \rho a_j^* u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) - 2s_{ij} - 2\rho u_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) \\ &\quad - \sigma_{ij} \left(\frac{2F''_{ij}(g_{ji}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} + 1 \right) - \sigma_{ji} \left(\frac{2F''_{ij}(g_{ij}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} + 1 \right). \end{aligned}$$

When $u_{ij}(\cdot)$ and $\tilde{u}_{ij}(\cdot)$ is sub-linear, we have

$$\frac{\Delta_{ij}\mathcal{L}}{\lambda \left(\tilde{f}_{ij} + \tilde{f}_{ji} \right)} \leq -2s_{ij} - \sigma_{ij} \left(\frac{2F''_{ij}(g_{ji}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} + 1 \right) - \sigma_{ji} \left(\frac{2F''_{ij}(g_{ij}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} + 1 \right). \quad (\text{C.52})$$

We will now show that at an optimal intervention (β, σ) , the link subsidy $\sigma_{ij}^* = \sigma_{ji}^* = 0$ when $s_{ij} \geq 0$. To establish a contradiction, suppose $\sigma_{ij}^* > 0$ and $\sigma_{ji}^* > 0$. By the KKT first-order

conditions, it must be that the derivative of the Lagrangian with respect to σ_{ij} and σ_{ji} is zero and thus $\Delta_{ij}\mathcal{L} = 0$. This is a contradiction to the inequality in (C.52).

Next, suppose $\sigma_{ij}^* > 0$ and $\sigma_{ji}^* = 0$. Analyzing the equilibrium link intensity condition in (C.34), because $\sigma_{ij}^* > \sigma_{ji}^*$ we have $g_{ij}^* > g_{ji}^*$. By assumption that the derivative of the cost of link intensity $F'_{ij}(\cdot)$ is weakly convex, the curvature

$$\tilde{f}_{ji} = \frac{1}{F''_{ij}(g_{ji}^*)} \geq \frac{1}{F''_{ij}(g_{ij}^*)} = \tilde{f}_{ij}.$$

We now compare the derivative of the Lagrangian with respect to σ_{ij} and σ_{ji} . By assumption that $\sigma_{ij}^* > 0$, the KKT first-order conditions gives $d\mathcal{L}/d\sigma_{ij} = 0$. Substituting in (C.48) we have

$$\tilde{f}_{ij} \underbrace{\left(\rho \left(\mathcal{R}'_i u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \mathcal{R}'_j u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) \right) - \lambda \left(\sigma_{ij}^* + \sigma_{ji}^* \right) \right)}_{\mathcal{T}_{ij}} = \lambda G_{ij}^*.$$

Since $g_{ij}^* > 0$, the overall link $G_{ij}^* > 0$ and consequently $\mathcal{T}_{ij} > 0$. Analyzing the derivative of the Lagrangian with respect to σ_{ji} in (C.50), we have

$$\frac{d\mathcal{L}}{d\sigma_{ji}} = \tilde{f}_{ji} \mathcal{T}_{ij} - \lambda G_{ij}^* \geq \tilde{f}_{ij} \mathcal{T}_{ij} - \lambda G_{ij}^* = \frac{d\mathcal{L}}{d\sigma_{ij}}.$$

Here, the inequality follows from the fact that $\mathcal{T}_{ij} > 0$ and $\tilde{f}_{ji} \geq \tilde{f}_{ij}$. Using $\frac{d\mathcal{L}}{d\sigma_{ij}} = 0$, we have $d\mathcal{L}/d\sigma_{ji} \geq 0$ and thus $\Delta_{ij}\mathcal{L} \geq 0$. But this is a contradiction to the bound derived in (C.52). A symmetric analysis establishes that it cannot be the case $\sigma_{ji}^* > 0$ and $\sigma_{ij}^* = 0$. Thus, it must be $\sigma_{ij}^* = \sigma_{ji}^* = 0$.

Proof of Part (ii): Suppose $\beta_i^* > 0$ and $\beta_j^* > 0$. By the KKT first-order conditions, $\mathcal{R}'_i = \lambda a_i^*$ and $\mathcal{R}'_j = \lambda a_j^*$. Substituting in (C.51), we have

$$\Delta_{ij}\mathcal{L} = \lambda \left(\tilde{f}_{ij} + \tilde{f}_{ji} \right) \left(\rho \left(a_i^* u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \rho a_j^* u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) \right) - \frac{2G_{ij}^*}{\tilde{f}_{ij} + \tilde{f}_{ji}} - \sigma_{ij} - \sigma_{ji} \right).$$

Suppose the cost $F_{ij}(\cdot)$ is sub-quadratic. Then, the equation can be bounded and rearranged

as

$$\begin{aligned} \Delta_{ij}\mathcal{L} \cdot \frac{1}{\lambda(\tilde{f}_{ij} + \tilde{f}_{ji})} &\geq \rho a_i^* u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \rho a_j^* u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) - 2F'_{ij}(g_{ij}^*) \frac{F''_{ij}(g_{ji}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} \\ &\quad - 2F'_{ij}(g_{ji}^*) \frac{F''_{ij}(g_{ij}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} - \sigma_{ij} - \sigma_{ji}, \end{aligned}$$

where we substituted $\tilde{f}_{ij} + \tilde{f}_{ji} = 1/F''_{ij}(g_{ij}^*) + 1/F''_{ij}(g_{ji}^*)$. Substituting the expression of equilibrium link first-order conditions (C.34) above, we have

$$\begin{aligned} \frac{\Delta_{ij}\mathcal{L}}{\lambda(\tilde{f}_{ij} + \tilde{f}_{ji})} &\geq \rho a_i^* u'_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) + \rho a_j^* u_{ij}(a_i^*) \tilde{u}'_{ij}(a_j^*) - 2s_{ij} - 2\rho u_{ij}(a_i^*) \tilde{u}_{ij}(a_j^*) \\ &\quad - \sigma_{ij} \left(\frac{2F''_{ij}(g_{ji}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} + 1 \right) - \sigma_{ji} \left(\frac{2F''_{ij}(g_{ij}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} + 1 \right). \quad (\text{C.53}) \end{aligned}$$

If $u_{ij}(\cdot)$ and $\tilde{u}_{ij}(\cdot)$ is super-linear, then

$$\frac{\Delta_{ij}\mathcal{L}}{\lambda(\tilde{f}_{ij} + \tilde{f}_{ji})} \geq -2s_{ij} - \sigma_{ij} \left(\frac{2F''_{ij}(g_{ji}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} + 1 \right) - \sigma_{ji} \left(\frac{2F''_{ij}(g_{ij}^*)}{F''_{ij}(g_{ji}^*) + F''_{ij}(g_{ij}^*)} + 1 \right).$$

If $s_{ij} < 0$ and $\sigma_{ij}^* = \sigma_{ji}^* = 0$, then by the above equation $\Delta_{ij}\mathcal{L} > 0$. This contradicts the KKT first-order conditions which imply $\Delta_{ij}\mathcal{L} = 0$. Thus, we must have $\sigma_{ij}^* + \sigma_{ji}^* > 0$. $\square$

Appendix D

Appendix for Robust Analysis of Winner-Pays-Bid Auctions With Large Supply

D.1 Omitted Proofs

D.1.1 Chernoff Bounds

This section presents a Chernoff bound which we utilize extensively to establish bounds on the allocation probability.

Fact 5. Let $X = \sum_{i=1}^n X_i$, where $X_i = 1$ with probability p_i and $X_i = 0$ with probability $1 - p_i$, and all X_i are independent. Let $\mu = \mathbb{E}(X) = \sum_{i=1}^n p_i$. Then

(i) **Upper Tail:** $\Pr(X \geq (1 + \lambda)\mu) \leq e^{-\frac{\lambda^2}{2+\lambda}\mu}$ for all $\lambda > 0$;

(ii) **Lower Tail:** $\Pr(X \leq (1 - \lambda)\mu) \leq e^{-\frac{\mu\lambda^2}{2}}$ for all $\lambda \in (0, 1)$.

D.1.2 Allocation Rule and k -th Order Statistic

This section presents a useful fact that relates the allocation probability of an agent with the distribution function of the k -th order statistic.

Fact 6. Consider any profile of bid distributions $\mathcal{E}$. For any bid $b \in \mathbb{R}_{\geq 0}$ and any agent $i \in \mathcal{N}$, the allocation probability and k -th order statistic satisfies

$$x_i(b + \Delta) \geq \mathcal{B}^{(k)}(b) \quad \text{for all } \Delta > 0.$$

Furthermore, if the distribution $\mathcal{E}_i$ for every agent i has no atoms, then

$$x_i(b) \geq \mathcal{B}^{(k)}(b).$$

Proof. Consider any $b \geq 0$. Suppose the k^{th} highest bid is at most b . Then, the k^{th} highest bid among agents other than i is strictly smaller than $b + \Delta$ for any $\Delta > 0$. Thus, we have

$$x_i(b + \Delta) \geq \mathcal{B}^{(k)}(b) \quad \text{for all } \Delta \geq 0.$$

Next, if no distribution $\mathcal{E}_i$ has atoms, ties occur with zero probability. Thus, if the k -th highest bid overall is strictly smaller than b , then the k -th highest bid among agents other than i is also strictly smaller than b . This implies $x_i(b) \geq \mathcal{B}^{(k)}(b)$. $\square$

D.1.3 Proof of **Proposition 13**

By definition of η -individual efficiency, the utility of the entrant agent with value v satisfies

$$u(v) + \widehat{B} \geq \eta v. \tag{D.1}$$

We denote the entrant agent's strategy by $\mathcal{E}_*$. Consider any participating agent $i \in \mathcal{N}$. Fix a $\Delta > 0$ and suppose agent i plays the bid $b + \Delta$ where $b \sim \mathcal{E}_*$. For any value v_i , the agent's

equilibrium utility satisfies

$$\begin{aligned} u_i(v_i) &\geq \mathbb{E}_{b \sim \mathcal{E}_*} [u_i(v_i, b + \Delta)] - \epsilon, \\ &\geq u(v_i) - \Delta - \epsilon, \end{aligned}$$

where the second inequality follows from [Fact 6](#). Substituting [\(D.1\)](#) in the inequality above, we get

$$u_i(v_i) + \hat{B} \geq \eta v_i - \Delta - \epsilon. \quad (\text{D.2})$$

Consider a valuation profile $\mathbf{v} = (v_1, v_2, \dots, v_{n(k)})$ and agent i . Suppose $\mathbf{y}^*(\mathbf{v})$ is a welfare maximizing allocation. Since $y_i^*(\mathbf{v}) \in [0, 1]$, we can multiply [\(D.2\)](#) by $y_i^*(\mathbf{v})$ to obtain

$$u_i(v_i) + \hat{B} y_i^*(\mathbf{v}) \geq \eta v_i y_i^*(\mathbf{v}) - \Delta y_i^*(\mathbf{v}) - \epsilon y_i^*(\mathbf{v}).$$

The feasibility constraint of the auction ensures that the total allocation does not exceed the supply, that is, $\sum_{i \in \mathcal{N}} y_i^*(\mathbf{v}) = k$. Substituting in the equation above gives

$$\sum_{i \in \mathcal{N}} u_i(v_i) + \hat{B} k \geq \eta \sum_{i \in \mathcal{N}} v_i y_i^*(\mathbf{v}) - k(\Delta + \epsilon). \quad (\text{D.3})$$

Now, the competitive efficiency of a multi-unit winner-pays-bid auction is 1, that is, the expected revenue $\mathbb{E}[\text{Revenue}(\mathbf{b})] \geq \mathbb{E}[\hat{B} \sum_{i \in \mathcal{N}^{(k)}} y_i^*(\mathbf{v})]$. Substituting this in [\(D.3\)](#) gives

$$\sum_{i \in \mathcal{N}^{(k)}} u_i(v_i) + \mathbb{E}[\text{Revenue}(\mathbf{b})] \geq \eta \sum_{i \in \mathcal{N}} v_i y_i^*(\mathbf{v}) - k(\Delta + \epsilon).$$

Taking an expectation with respect to the value distributions $\mathbf{V} = (\mathcal{V}_i)_{i \in \mathcal{N}^{(k)}}$ gives

$$\frac{\mathcal{W}(\mathcal{E}, \mathcal{A})}{\mathcal{O}(\mathcal{A})} \geq \eta - (\epsilon + \Delta) \cdot \frac{k}{\mathcal{O}(\mathcal{A})}.$$

Taking the limit $\Delta \rightarrow 0$ establishes the statement of the result.

D.1.4 Proof of Proposition 14

Fix any $c > 0$. By assumption, let $\left\{ \left(\underline{b}^{(k)}, \bar{b}^{(k)} \right) \right\}_{k=1}^{\infty}$ be the extremizing sequence of bid pairs and $\left(\mathcal{A}^{(k_l)} \right)_{l=1}^{\infty}$ the corresponding subsequence of auctions satisfying

$$\lim_{l \rightarrow \infty} \int_{\underline{b}^{(k_l)}}^{\bar{b}^{(k_l)}} \mathcal{B}^{(k_l)}(b) db \leq c.$$

For the rest of the proof, we suppress the subsequence index to simplify notation. Thus, whenever we take limits, they are to be understood as limits along the subsequence of auctions $\left(\mathcal{A}^{(k_l)} \right)_{l=1}^{\infty}$.

Consider any agent $i \in \mathcal{N}^{(k)}$ and value v_i . Furthermore, consider a sequence of perturbation parameters $\left(\Delta^{(k)} \right)_{k=1}^{\infty}$ such that $\Delta^{(k)} > 0$ and $\lim_{k \rightarrow \infty} \Delta^{(k)} = 0$. Suppose the agent bids $\bar{b}^{(k)} + \Delta^{(k)}$. Let $u_i(v_i)$ denote the agents equilibrium utility and $u_i(v_i, b)$ denote the utility obtained from bidding b . Since the strategy profile $\mathcal{E}^{(k)}$ is an ϵ_k -approximate equilibrium, we have

$$u_i(v_i) \geq u_i \left(v_i, \bar{b}^{(k)} + \Delta^{(k)} \right) - \epsilon_k, \quad (\text{D.4})$$

$$= \left(v_i - \bar{b}^{(k)} - \Delta^{(k)} \right) x_i \left(\bar{b}^{(k)} + \Delta^{(k)} \right) - \epsilon_k, \quad (\text{D.5})$$

$$\geq \left(v_i - \bar{b}^{(k)} - \Delta^{(k)} \right) \mathcal{B}^{(k)} \left(\bar{b}^{(k)} \right) - \epsilon_k. \quad (\text{D.6})$$

Here, the second inequality follows from Fact 6.

We now solve for the expectation of $\hat{B}$. By the tail sum formula for expectations,

$$\hat{B} = \int_0^{\infty} (1 - \mathcal{B}^{(k)}(b)) db.$$

By monotonicity of the distribution function $\mathcal{B}^{(k)}(\cdot)$, for any bid $b \leq \underline{b}^{(k)}$ we have $\mathcal{B}^{(k)}(b) \leq \mathcal{B}^{(k)}(\underline{b}^{(k)})$. Then, a lower bound on the expectation is

$$\hat{B} \geq \left(1 - \mathcal{B}^{(k)}(\underline{b}^{(k)}) \right) \underline{b}^{(k)} + \left(\bar{b}^{(k)} - \underline{b}^{(k)} \right) - \int_{\underline{b}^{(k)}}^{\bar{b}^{(k)}} \mathcal{B}^{(k)}(b) db. \quad (\text{D.7})$$

Combining (D.6) and (D.7), we have

$$u_i(v_i) + \widehat{B} \geq v_i \mathcal{B}^{(k)}(\bar{b}^{(k)}) - \underline{b}^{(k)} \mathcal{B}^{(k)}(\underline{b}^{(k)}) - \epsilon_k - \Delta^{(k)} \mathcal{B}^{(k)}(\bar{b}^{(k)}) - \int_{\underline{b}^{(k)}}^{\bar{b}^{(k)}} \mathcal{B}^{(k)}(b) db. \quad (\text{D.8})$$

Consider a valuation profile $\mathbf{v} = (v_1, v_2, \dots, v_{n^{(k)}})$ and agent i . Suppose $\mathbf{y}^*(\mathbf{v})$ is a welfare maximizing allocation. Since $y_i^*(\mathbf{v}) \in [0, 1]$, we can multiply (D.8) by $y_i^*(\mathbf{v})$ to obtain

$$\begin{aligned} u_i(v_i) + \widehat{B} y_i^*(\mathbf{v}) &\geq \mathcal{B}^{(k)}(\bar{b}^{(k)}) v_i y_i^*(\mathbf{v}) - y_i^*(\mathbf{v}) \int_{\underline{b}^{(k)}}^{\bar{b}^{(k)}} \mathcal{B}^{(k)}(b) db - \Delta^{(k)} \mathcal{B}^{(k)}(\bar{b}^{(k)}) y_i^*(\mathbf{v}) \\ &\quad - \underline{b}^{(k)} \mathcal{B}^{(k)}(\underline{b}^{(k)}) y_i^*(\mathbf{v}) - y_i^*(\mathbf{v}) \epsilon_k. \end{aligned}$$

The feasibility constraint of the auction ensures that the total allocation does not exceed the supply, i.e., $\sum_{i \in \mathcal{N}^{(k)}} y_i^*(\mathbf{v}) = k$. Moreover, each bid $\underline{b}^{(k)}$ is bounded above by the bound on values H . Substituting this in the equation above gives

$$\begin{aligned} \sum_{i \in \mathcal{N}^{(k)}} u_i(v_i) + \widehat{B} \sum_{i \in \mathcal{N}^{(k)}} y_i^*(\mathbf{v}) &\geq \mathcal{B}^{(k)}(\bar{b}^{(k)}) \left(\sum_{i \in \mathcal{N}^{(k)}} v_i y_i^*(\mathbf{v}) \right) - k \int_{\underline{b}^{(k)}}^{\bar{b}^{(k)}} \mathcal{B}^{(k)}(b) db \\ &\quad - H k \mathcal{B}^{(k)}(\underline{b}^{(k)}) - k \epsilon_k - k \Delta^{(k)} \mathcal{B}^{(k)}(\bar{b}^{(k)}). \quad (\text{D.9}) \end{aligned}$$

Now, the competitive efficiency of a multi-unit winner-pays-bid auction is one, that is, the expected revenue $\mathbb{E}[\text{Revenue}(\mathbf{b})] \geq \mathbb{E}[\widehat{B} \sum_{i \in \mathcal{N}^{(k)}} y_i^*(\mathbf{v})]$. Substituting this in (D.9) gives

$$\begin{aligned} \sum_{i \in \mathcal{N}^{(k)}} u_i(v_i) + \mathbb{E}[\text{Revenue}(\mathbf{b})] &\geq \mathcal{B}^{(k)}(\bar{b}^{(k)}) \left(\sum_{i \in \mathcal{N}^{(k)}} v_i y_i^*(\mathbf{v}) \right) - k \int_{\underline{b}^{(k)}}^{\bar{b}^{(k)}} \mathcal{B}^{(k)}(b) db \\ &\quad - H k \mathcal{B}^{(k)}(\underline{b}^{(k)}) - k \epsilon_k - k \Delta^{(k)}. \quad (\text{D.10}) \end{aligned}$$

Taking an expectation with respect to the value distributions $\mathbf{v} = (\mathcal{V}_i)_{i \in \mathcal{N}^{(k)}}$ gives

$$\frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq \mathcal{B}^{(k)}(\bar{b}^{(k)}) - \frac{k \cdot \int_{\underline{b}^{(k)}}^{\bar{b}^{(k)}} \mathcal{B}^{(k)}(b) db + H k \mathcal{B}^{(k)}(\underline{b}^{(k)}) + k \epsilon_k + k \Delta^{(k)}}{\mathcal{O}(\mathcal{A}^{(k)})}.$$

By [Assumption 4](#), there exists a constant $\zeta > 0$ such that $\mathcal{O}(\mathcal{A}^{(k)}) \geq k\zeta$. It follows that

$$\frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq \mathcal{B}^{(k)}(\bar{b}^{(k)}) - \frac{1}{\zeta} \int_{\underline{b}^{(k)}}^{\bar{b}^{(k)}} \mathcal{B}^{(k)}(b) db - \frac{H}{\zeta} \mathcal{B}^{(k)}(\underline{b}^{(k)}) - \frac{\epsilon_k}{\zeta} - \frac{\Delta^{(k)}}{\zeta}. \quad (\text{D.11})$$

By assumption, the sequence of bid pairs is extremizing. Next, along subsequence $(\mathcal{A}^{(k_l)})_{l=1}^\infty$, the limit of the integral is

$$\lim_{l \rightarrow \infty} \int_{\underline{b}^{(k_l)}}^{\bar{b}^{(k_l)}} \mathcal{B}^{(k_l)}(b) db < c.$$

Finally, the perturbation $\Delta^{(k)}$ converges to 0. Thus, substituting in [\(D.11\)](#), the limit of the ratio is

$$\lim_{l \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k_l)}, \mathcal{A}^{(k_l)})}{\mathcal{O}(\mathcal{A}^{(k_l)})} \geq 1 - \frac{c}{\zeta} - \frac{\epsilon^*}{\zeta}.$$

The statement of the result is proved.

D.1.5 Proof of [Theorem 7](#)

In this section, we prove a sequence of results that are useful to prove [Theorem 7](#). For the analysis, it is without loss to make the following assumption on equilibrium bid distributions.

Assumption 9. Consider any sequence of auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$. The corresponding sequence of equilibria $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$ are such that the bid distribution $\mathcal{E}_i^{(k)}$ for every agent $i \in \mathcal{N}^{(k)}$ and $k \in \mathbb{N}$ has no atoms.

Lemma 15. Suppose [Assumption 9](#) is satisfied. Consider any sequence of bid pairs $\left\{(\underline{b}^{(k)}, \bar{b}^{(k)})\right\}_{k=1}^\infty$ where $\underline{b}^{(k)} \in [0, \underline{\psi}_*^{(k)}]$ and $\bar{b}^{(k)} \in [\bar{\psi}_*^{(k)}, \infty)$. The sequence is extremizing over the sequence of auctions $(\mathcal{A}^{(k)})_{k=1}^\infty$.

Proof. Consider any $k \in \mathbb{N}$. We begin by analyzing the distribution of the k^{th} highest bid $\mathcal{B}^{(k)}$ at the higher bid $\bar{b}^{(k)}$. This involves solving for the probability that at most $k-1$

equilibrium bids are strictly above $\bar{\psi}_*^{(k)}$. Consider the following sequence of inequalities:

$$\begin{aligned} \mathbb{P} \left[\mathcal{T} \left(\bar{\psi}_*^{(k)} \right) \leq k - 1 \right] &= \mathbb{P} \left[\mathcal{T} \left(\bar{\psi}_*^{(k)} \right) - \mathbb{E} \left[\mathcal{T} \left(\bar{\psi}_*^{(k)} \right) \right] \leq k - 1 - \mathbb{E} \left[\mathcal{T} \left(\bar{\psi}_*^{(k)} \right) \right] \right], \\ &\geq \mathbb{P} \left[\mathcal{T} \left(\bar{\psi}_*^{(k)} \right) - \mathbb{E} \left[\mathcal{T} \left(\bar{\psi}_*^{(k)} \right) \right] \leq k^{0.5+\delta} \right], \\ &\geq 1 - 2e^{-\frac{1}{3} \cdot \frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}}. \end{aligned}$$

The first inequality follows when we substitute $\mathbb{E} \left[\mathcal{T} \left(\bar{\psi}_*^{(k)} \right) \right] = k - k^{0.5+\delta} - 1$, which holds by definition of $\bar{\psi}_*^{(k)}$ and by assumption that no bid distribution has atoms. The final inequality follows from applying the upper tail Chernoff bound in [Fact 5](#) to $\mathcal{T} \left(\bar{\psi}_*^{(k)} \right)$ — a sum of independent indicator random variables.

We begin by analyzing the probability of the following event: the k^{th} highest equilibrium bid is at most $\bar{b}^{(k)}$. This event occurs if the total equilibrium bids strictly above $\bar{\psi}_*^{(k)}$ is at most $(k - 1)$. Thus, the distribution function at $\bar{b}^{(k)}$ is at least

$$\mathbb{P} \left[\mathcal{B}^{(k)} \leq \bar{b}^{(k)} \right] \geq 1 - 2e^{-\frac{1}{3} \cdot \frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}},$$

and taking the limit $k \rightarrow \infty$ implies that $\lim_{k \rightarrow \infty} \mathbb{P} \left[\mathcal{B}^{(k)} \leq \bar{b}^{(k)} \right] = 1$.

Next, we analyze the distribution $\mathcal{B}^{(k)}$ at the lower bid $\underline{b}^{(k)}$. Suppose $\underline{\psi}_*^{(k)} > 0$. By definition of $\underline{\psi}_*^{(k)}$ and by assumption that equilibrium bids have no atoms, the expected number of equilibrium bids above $\underline{b}^{(k)}$ satisfies $\mathbb{E} \left[\mathcal{T} \left(\underline{b}^{(k)} \right) \right] = k + k^{0.5+\delta} - 1$. We will use this to establish a concentration inequality on the distribution of the k -th highest bid $\mathcal{B}^{(k)}$.

By monotonicity of the distribution function, we have $\mathbb{P} \left[\mathcal{B}^{(k)} \leq \underline{b}^{(k)} \right] \leq \mathbb{P} \left[\mathcal{B}^{(k)} \leq \underline{\psi}_*^{(k)} \right]$. The k^{th} highest equilibrium bid is at most $\underline{\psi}_*^{(k)}$, then at most $(k - 1)$ bids are strictly above $\underline{\psi}_*^{(k)}$.

Consider the following sequence of inequalities:

$$\mathbb{P} \left[\mathcal{T} \left(\underline{\psi}_*^{(k)} \right) \leq k - 1 \right] = \mathbb{P} \left[\mathcal{T} \left(\underline{\psi}_*^{(k)} \right) - \mathbb{E} \left[\mathcal{T} \left(\underline{\psi}_*^{(k)} \right) \right] \leq k - 1 - \mathbb{E} \left[\mathcal{T} \left(\underline{\psi}_*^{(k)} \right) \right] \right], \quad (\text{D.12})$$

$$= \mathbb{P} \left[\mathcal{T} \left(\underline{\psi}_*^{(k)} \right) - \mathbb{E} \left[\mathcal{T} \left(\underline{\psi}_*^{(k)} \right) \right] \leq -k^{0.5+\delta} \right], \quad (\text{D.13})$$

$$\leq 2e^{-\frac{1}{3} \cdot \frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}}. \quad (\text{D.14})$$

The inequality follows from applying the lower tail Chernoff bound in [Fact 5](#). Thus, the distribution function at $\underline{b}^{(k)}$ is at most

$$\mathbb{P} \left[\mathcal{B}^{(k)} \leq \underline{b}^{(k)} \right] \leq 2e^{-\frac{1}{3} \cdot \frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}}.$$

Now, suppose $\underline{\psi}_*^{(k)} = 0$. Since no distribution has an atom, $\mathbb{P} \left[\mathcal{B}^{(k)} \leq 0 \right] = 0$. Taking the limit $k \rightarrow \infty$ gives $\lim_{k \rightarrow \infty} \mathbb{P} \left[\mathcal{B}^{(k)} \leq \underline{b}^{(k)} \right] = 0$ and the statement of the result follows. $\square$

The proof utilizes a Chernoff bound to establish that the constructed bid pairs are extremizing. We provide some intuition for this result. By definition, the expected number of equilibrium bids strictly above $\overline{\psi}_*^{(k)}$ — and hence above any bid $\bar{b}^{(k)} > \overline{\psi}_*^{(k)}$ — is at most $k - k^{0.5+\delta} - 1$. As k becomes large, the probability that more than k bids exceed $\bar{b}^{(k)}$ is negligible. Conversely, for any $\underline{b}^{(k)} < \underline{\psi}_*^{(k)}$, the expected number of bids strictly above it is at least $k + k^{0.5+\delta} - 1$, implying that the probability of observing at least k such bids converges to one.

We have established the existence and provided a construction of an extremizing sequence of bid pairs. In order to apply [??](#), it remains to establish that there exists an extremizing sequence of bid pairs that converge to one another, that is,

$$\lim_{k \rightarrow \infty} \left(\bar{b}^{(k)} - \underline{b}^{(k)} \right) = 0.$$

Before that, we prove a useful result which shows that the allocation probability for any agent at any bid strictly smaller than $\underline{\psi}_*^{(k)}$ vanishes. This conclusion is not immediate from [Lemma 15](#) because the k -th order statistic is only a lower bound on the allocation probability

(see Fact 6).

Lemma 16. *Suppose equilibrium bid distributions $\mathcal{E}^{(k)}$ satisfy Assumption 9. Consider any sequence of bids $\left(\underline{b}^{(k)}\right)_{k=1}^{\infty}$ satisfying $\underline{b}^{(k)} \in \left[0, \underline{\psi}^{(k)}\right]$. The allocation probability of any agent $i \in \mathcal{N}^{(k)}$ at bid $\underline{b}^{(k)}$ converges to 0 as $k \rightarrow \infty$, that is,*

$$\lim_{k \rightarrow \infty} \max_{i \in \mathcal{N}^{(k)}} x_i^{(k)} \left(\underline{b}^{(k)} \right) = 0.$$

Proof. We begin by analyzing the case $\underline{\psi}^{(k)} > 0$. By assumption that equilibrium bid distributions have a continuous distribution function, it must be $\mathbb{E} \left[\mathcal{T} \left(\underline{\psi}^{(k)} \right) \right] = k + k^{0.5+\delta} - 1$. Consequently, for every bid $\underline{b}^{(k)} \in \left[0, \underline{\psi}^{(k)}\right]$, the expected number of equilibrium bids above $\underline{b}^{(k)}$ is

$$\mathbb{E} \left[\mathcal{T} \left(\underline{b}^{(k)} \right) \right] \geq k + k^{0.5+\delta} - 1. \quad (\text{D.15})$$

We now analyze the allocation probability of an individual i at bid $\underline{b}^{(k)}$. This involves analyzing the distribution function of their critical bid. To that end, we define

$$\mathcal{T}_{-i}(b) := \sum_{l \in \mathcal{N}^{(k)} \setminus \{i\}} \mathcal{T}_l(b),$$

as the total competing equilibrium bids to agent i that are strictly above b . Since $\mathcal{T}_i \left(\underline{b}^{(k)} \right)$ is an indicator random variable, it follows from (D.15) that

$$\mathbb{E} \left[\mathcal{T}_{-i} \left(\underline{b}^{(k)} \right) \right] \geq k + k^{0.5+\delta} - 2. \quad (\text{D.16})$$

We solve for the allocation probability $x_i^{(k)} \left(\underline{b}^{(k)} \right)$. Individual i is allocated at bid $\underline{b}^{(k)}$ only if the total number of competing equilibrium bids above $\underline{b}^{(k)}$ is at most $(k-1)$. This gives the following bound on the allocation probability:

$$x_i^{(k)} \left(\underline{b}^{(k)} \right) \leq \mathbb{P} \left[\mathcal{T}_{-i} \left(\underline{b}^{(k)} \right) \leq k-1 \right].$$

Consider the following chain of inequalities that derive a bound for the right-hand side in the expression above:

$$x_i^{(k)}(\underline{b}^{(k)}) \leq \mathbb{P} \left[\mathcal{T}_{-i}(\underline{b}^{(k)}) \leq k-1 \right], \quad (\text{D.17})$$

$$\leq \mathbb{P} \left[\mathcal{T}_{-i}(\underline{b}^{(k)}) - \mathbb{E} \left[\mathcal{T}_{-i}(\underline{b}^{(k)}) \right] \leq k-1 - \mathbb{E} \left[\mathcal{T}_{-i}(\underline{b}^{(k)}) \right] \right], \quad (\text{D.18})$$

$$\leq e^{-\frac{1}{2} \cdot \frac{(\mathbb{E}[\mathcal{T}_{-i}(\underline{b}^{(k)})] - k + 1)^2}{\mathbb{E}[\mathcal{T}_{-i}(\underline{b}^{(k)})]}}, \quad (\text{D.19})$$

$$\leq e^{-\frac{1}{2} \cdot \frac{k^{2\delta} - \frac{2}{k^{0.5-\delta}} + \frac{1}{k}}{1 + \frac{1}{k^{0.5-\delta}} - \frac{2}{k}}}. \quad (\text{D.20})$$

The second inequality follows from applying the lower tail Chernoff bound in [Fact 5](#). To derive an upper bound on (D.19), it suffices to analyze the function in the exponent, rewritten below for convenience:

$$\frac{(\mathbb{E}[\mathcal{T}_{-i}(\underline{b}^{(k)})] - k + 1)^2}{\mathbb{E}[\mathcal{T}_{-i}(\underline{b}^{(k)})]}.$$

For a fixed k , the term above is strictly increasing in the region $\mathbb{E}[\mathcal{T}_{-i}(\underline{b}^{(k)})] > k-1$. Utilizing the inequality in (D.1.5), an upper bound to (D.13) is obtained by substituting $\mathbb{E}[\mathcal{T}(\underline{b}^{(k)})] = k + k^{0.5+\delta} - 2$.

Next, suppose $\psi_*^{(k)} = 0$. By assumption that no distributions have an atom, the allocation probability at a zero bid is zero. Combining this observation with (D.20), we get

$$x_i^{(k)}(\underline{b}^{(k)}) \leq e^{-\frac{1}{2} \cdot \frac{k^{2\delta} - \frac{2}{k^{0.5-\delta}} + \frac{1}{k}}{1 + \frac{1}{k^{0.5-\delta}} - \frac{2}{k}}}.$$

Moreover, this is satisfied for every agent $i \in \mathcal{N}^{(k)}$. Thus $\lim_{k \rightarrow \infty} \max_{i \in \mathcal{N}^{(k)}} x_i^{(k)}(\underline{b}^{(k)}) = 0$ and the statement of the result is established. $\square$

Assumption 10. *The sequence of bid profiles $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$ is such that $\bar{\psi}_*^{(k)} < H$ for every $k \in \mathbb{N}$.*

Lemma 17. Consider any sequence of k -unit first-price auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$ and a corresponding sequence of ϵ -approximate Bayes-Nash equilibrium $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$ satisfying [Assumption 10](#). Suppose there exists a $c > 0$ such that for any extremizing sequence of bid pairs $\left\{ \left(\underline{b}^{(k)}, \bar{b}^{(k)} \right) \right\}_{k=1}^{\infty}$ and any corresponding subsequence of auctions $(\mathcal{A}^{(k_l)})_{l=1}^{\infty}$ along which the bid difference converges, we have

$$\lim_{l \rightarrow \infty} \left(\bar{b}^{(k_l)} - \underline{b}^{(k_l)} \right) > c.$$

Then, there exists a constant $\bar{u} > 0$, a subsequence $(\mathcal{A}^{(k_m)})_{m=1}^{\infty}$ and $m^* \in \mathbb{N}$ such that for all $m > m^*$, there exist an interval

$$\mathcal{W}^{(k_m)} \subset \left(\underline{\psi}_*^{(k_m)}, \bar{\psi}_*^{(k_m)} \right)$$

such that

- (a) the length of the interval denoted by $\text{len}(\mathcal{W}^{(k_m)}) \geq c/8$, and,
- (b) for every agent $i \in \mathcal{N}^{(k_m)}$, every value $v \in \mathcal{W}^{(k_m)}$ yields optimal utility at least $\bar{u}$.

Proof. By [Lemma 15](#), the bid pair sequence $\left\{ \left(\underline{\psi}_*^{(k)}, \bar{\psi}_*^{(k)} \right) \right\}_{k=1}^{\infty}$ is extremizing along the sequence of auctions $(\mathcal{A}^{(k)})_{k=1}^{\infty}$. Furthermore, by [Assumption 10](#), since $\underline{\psi}_*^{(k)}$ and $\bar{\psi}_*^{(k)}$ lie in the compact interval $[0, H]$, there must exist a subsequence of auctions along which the difference $\bar{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}$ converges. By assumption, it must be

$$\lim_{l \rightarrow \infty} \left(\bar{\psi}_*^{(k_l)} - \underline{\psi}_*^{(k_l)} \right) > c, \tag{D.21}$$

where l is the index of any subsequence of auctions along which the bid difference converges. We will fix one such subsequence and analyze this for the rest of the proof.

We now establish that a constant length subset of values in the interval $\left(\underline{\psi}_*^{(k_l)}, \bar{\psi}_*^{(k_l)} \right)$ obtain strictly positive optimal utility for k_l large enough. We prove this by constructing a sequence of bids within the interval having a strictly positive distribution function in the limit. To

that end, consider a sequence of bids $(b^{(k_l)})_{k=1}^\infty$ with $b^{(k_l)}$ defined as

$$b^{(k_l)} := \max \left\{ \bar{\psi}_*^{(k_l)} - \frac{c}{2}, 0 \right\}. \quad (\text{D.22})$$

Note that for such a subsequence, $b^{(k_l)}$ is positive for l large enough.

To establish a contradiction, suppose there exists a further subsequence of $(\mathcal{A}^{(k_l)})_{l=1}^\infty$, which we denote by $(\mathcal{A}^{(k_m)})_{m=1}^\infty$, such that the following is satisfied:

- (a) The limit of the bid difference $\bar{\psi}_*^{(k_m)} - b^{(k_m)}$ converges.
- (b) The limit of the distribution function at $b^{(k_m)}$ satisfies $\lim_{m \rightarrow \infty} \mathcal{B}^{(k_m)}(b^{(k_m)}) = 0$.

The existence of a subsequence along which the bid difference converges follows because $\bar{\psi}_*^{(k_l)}$ and $b^{(k_l)}$ lie in a compact interval. Next, since $\mathcal{B}^{(k_l)}(b^{(k_l)})$ lies in the compact interval $[0, 1]$ for every $l \in \mathbb{N}$, there exists a further subsequence along which the limit of the distribution function is well defined.

As established in the proof of [Lemma 15](#), $\lim_{m \rightarrow \infty} \mathcal{B}^{(k_m)}(\bar{\psi}_*^{(k_m)}) = 1$. By definition, the sequence of bid pairs $\left\{ (b^{(k_m)}, \bar{\psi}_*^{(k_m)}) \right\}$ is extremizing along the subsequence of auctions $(\mathcal{A}^{(k_m)})_{m=1}^\infty$. However, the limit of the bid difference is $\lim_{m \rightarrow \infty} (\bar{\psi}_*^{(k_m)} - b^{(k_m)}) = c/2$. This is a contradiction to the assumption that for any extremizing sequence of bid pairs and any subsequence along which the bid difference converges, the limit of the bid difference is at least c . Thus, it must be that

$$\lim_{m \rightarrow \infty} \mathcal{B}^{(k_m)}(b^{(k_m)}) > 0. \quad (\text{D.23})$$

Consider any agent $i \in \mathcal{N}^{(k)}$ and value $v \in \left[\bar{\psi}_*^{(k)} - \frac{c}{4}, \bar{\psi}_*^{(k)} - \frac{c}{8} \right]$. Along the subsequence of auctions $(\mathcal{A}^{(k_m)})_{m=1}^\infty$, for m large enough, the optimal interim utility at value v is at least

$$u_i^{(k_m)}(v) \geq (v - b^{(k_m)}) \cdot x_i^{(k_m)}(b^{(k_m)}) \geq \frac{c}{4} \cdot \mathcal{B}^{(k_m)}(b^{(k_m)}).$$

Here, the second inequality follows from the definition of $b^{(k_m)}$ in [\(D.22\)](#) and observing that

for large enough m , the bid $b^{(k_m)} = \bar{\psi}_*^{(k_m)} - c/2$. Furthermore, from Fact 6, we substitute $x_i^{(k_m)}(b^{(k_m)}) \geq \mathcal{B}^{(k_m)}(b^{(k_m)})$. Using (D.23) and taking the limit of $m \rightarrow \infty$, the optimal utility for any agent is strictly positive for m large enough. The statement of the result follows. $\square$

Lemma 18. *Consider any sequence of k -unit first-price auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$ and a corresponding sequence of $\epsilon^{(k)}$ -approximate Bayes-Nash equilibrium $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$ satisfying Assumption 10. Let $\epsilon^* := \lim_{k \rightarrow \infty} \epsilon^{(k)}$ and suppose $\epsilon^* > 0$. Then,*

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq 1 - \zeta' \epsilon \left(1 + \log \left(\frac{H}{\epsilon} \right) \right),$$

where $\zeta' > 0$ depends only on the density lower bound κ .

Proof. Consider any subsequence of auctions. We will overload notation and denote the index of this subsequence by k .

Suppose there exists an extremizing sequence of bid pairs $\{\underline{b}^{(k)}, \bar{b}^{(k)}\}$ and a corresponding subsequence $(\mathcal{A}_{k_l})_{l=1}^\infty$ along which the bid difference converges as

$$\lim_{l \rightarrow \infty} (\bar{b}^{(k_l)} - \underline{b}^{(k_l)}) \leq \epsilon.$$

The integral of the allocation rule in the interval $(\underline{b}^{(k_l)}, \bar{b}^{(k_l)})$ is

$$\int_{\underline{b}^{(k_l)}}^{\bar{b}^{(k_l)}} \mathcal{B}^{(k_l)}(b) db \leq (\bar{b}^{(k_l)} - \underline{b}^{(k_l)}).$$

By Proposition 14, the limit of the ratio of equilibrium welfare to optimal welfare along this subsequence is

$$\lim_{l \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k_l)}, \mathcal{A}^{(k_l)})}{\mathcal{O}(\mathcal{A}^{(k_l)})} \geq 1 - \frac{2\epsilon}{\zeta}. \quad (\text{D.24})$$

Next, suppose that for every extremizing sequence of bid pairs and any corresponding sub-

sequence of auctions along which the bid difference converges, we have

$$\lim_{l \rightarrow \infty} \left(\bar{b}^{(k_l)} - \underline{b}^{(k_l)} \right) > \epsilon.$$

By Lemma 17, there exists a subsequence of auctions $(\mathcal{A}^{(k_m)})_{m=1}^{\infty}$ such that for every m large enough, there is a compact interval $\mathcal{W}^{(k_m)} \subset \left(\underline{\psi}_*^{(k_m)}, \bar{\psi}_*^{(k_m)} \right)$ and a constant $\bar{u} > 0$ such that

(a) $\text{len}(\mathcal{W}^{(k_m)}) \geq \epsilon/8$, and,

(b) for every agent $i \in \mathcal{N}^{(k_m)}$, every value $v \in \mathcal{W}^{(k_m)}$ yields optimal utility at least $\bar{u}$.

Furthermore, as constructed in the proof of Lemma 17, an interval that satisfies these properties is $\left[\bar{\psi}_*^{(k_m)} - \epsilon/4, \bar{\psi}_*^{(k_m)} - \epsilon/8 \right]$. We will focus our attention on this particular $\mathcal{W}^{(k_m)}$ for the rest of the proof.

We now introduce some notation that will be used in the rest of the proof. Let $\mathcal{E}_{i,v}^{(k)}$ denote the equilibrium bid distribution of agent i for value v in auction $\mathcal{A}^{(k)}$. For values in the interval $\mathcal{W}^{(k)}$, we will compare the measure of equilibrium bids in the following two sets: $\underbrace{\left[0, \underline{\psi}_*^{(k)} \right] \cup \left(\bar{\psi}_*^{(k)}, \infty \right)}_{\mathcal{S}_*^{(k)}}$ and $\underbrace{\left(\underline{\psi}_*^{(k)}, \bar{\psi}_*^{(k)} \right]}_{\mathcal{M}_*^{(k)}}$. Finally, for value v we will define

$$\mathcal{R}_{i,v}^{(k)} := \mathcal{E}_{i,v}^{(k)}(\mathcal{S}_*^{(k)}) \quad \text{and} \quad \mathcal{T}_{i,v}^{(k)} := \mathcal{E}_{i,v}^{(k)}(\mathcal{M}_*^{(k)}),$$

as the measure of equilibrium bids in the interval $\mathcal{S}_*^{(k)}$ and $\mathcal{M}_*^{(k)}$ respectively. For any value v , we have $\mathcal{R}_{i,v}^{(k)} + \mathcal{T}_{i,v}^{(k)} = 1$.

We analyze the following quantities:

$$\mathcal{R}_*^{(k)} := \max_{i \in \mathcal{N}^{(k)}} \sup_{v \in \mathcal{W}^{(k)}} \mathcal{R}_{i,v}^{(k)} \quad \text{and} \quad \mathcal{T}_*^{(k)} := \min_{i \in \mathcal{N}^{(k)}} \inf_{v \in \mathcal{W}^{(k)}} \mathcal{T}_{i,v}^{(k)}. \quad (\text{D.25})$$

$\mathcal{R}_*^{(k)}$ is the maximum measure of equilibrium bids an agent and a particular value in $\mathcal{W}^{(k)}$ have in the interval $\mathcal{S}_*^{(k)}$. On the other hand, $\mathcal{T}_*^{(k)}$ is the minimum measure of equilibrium bids that any agent and any value in $\mathcal{W}^{(k)}$ has in the interval $\mathcal{M}_*^{(k)}$.

Consider any subsequence of $(k_m)_{m=1}^\infty$ along which the limit of $\mathcal{R}_*^{(k_m)}$ exists. We will denote such a subsequence with the index $(k_p)_{p=1}^\infty$. We will show that $\lim_{p \rightarrow \infty} \mathcal{R}_*^{(k_p)} = 1$. That is, as p increases, there exists an agent and a corresponding value which places equilibrium bid measure arbitrarily close to one in $\mathcal{S}_*^{(k_p)}$. Before we provide a proof, note that since $\mathcal{R}_*^{(k_m)}$ lies in the compact interval $[0, 1]$, there always exists a subsequence along which the limit of $\mathcal{R}_*^{(k_m)}$ exists.

To establish a contradiction, consider any subsequence along which the limit exists and $\lim_{p \rightarrow \infty} \mathcal{R}_*^{(k_p)} < 1$. Using $\mathcal{R}_{i,v}^{(k)} + \mathcal{T}_{i,v}^{(k)} = 1$, we have

$$\lim_{p \rightarrow \infty} \mathcal{T}_*^{(k_p)} > 0.$$

Consequently, there exists a constant $\beta > 0$ such that for p large enough, every agent $i \in \mathcal{N}^{(p)}$ and every value $v \in \mathcal{W}^{(k_p)}$ has at least a β measure of equilibrium bids in the interval $\mathcal{M}_*^{(k_p)}$. Combining [Assumption 4](#) (b) and observation (a) in [Lemma 17](#) — that $\text{len}(\mathcal{W}^{(k_p)}) \geq c/8$ — there exists a constant $\zeta > 0$ such that the total expected equilibrium bids in $\mathcal{M}_*^{(k_p)}$ is at least $\zeta n^{(k_p)}$ (recall, $n^{(k_p)}$ is the number of agents in $\mathcal{N}^{(k_p)}$).

We establish below that this is a contradiction to the definition of $\underline{\psi}_*^{(k_m)}$ and $\overline{\psi}_*^{(k_m)}$. By definition and [Assumption 9](#),

$$\mathbb{E} \left[\mathcal{T} \left(\overline{\psi}_*^{(k_m)} \right) \right] = k^{(m)} - (k^{(m)})^{0.5+\delta} - 1, \text{ and, } \mathbb{E} \left[\mathcal{T} \left(\underline{\psi}_*^{(k_m)} \right) \right] \leq k^{(m)} + (k^{(m)})^{0.5+\delta} - 1.$$

The total expected equilibrium bids in the interval $\left(\underline{\psi}_*^{(k_m)}, \overline{\psi}_*^{(k_m)} \right]$ is at most $2(k^{(m)})^{0.5+\delta}$. A contradiction to the conclusion in the paragraph above that for p large enough, there exists a constant $\zeta > 0$ such that the expected number of equilibrium bids in $\mathcal{W}^{(k_p)}$ is at least $\zeta n^{(k_p)}$. Thus, for any subsequence of $(k_m)_{m=1}^\infty$ along which the limit of $\mathcal{R}_*^{(k_m)}$ exists, we have $\lim_{p \rightarrow \infty} \mathcal{R}_*^{(k_p)} = 1$.

Consequently, there is a sequence of agent-value pairs $\left(\{i^{(k_p)}, v_{i^{(k_p)}}\} \right)_{p=1}^\infty$ with the following properties:

- (a) for every $p \in \mathbb{N}$, the value $v_{i^{(k_p)}} \in \mathcal{W}^{(k_p)}$, and,

(b) the limit of the measure of equilibrium bids in $\mathcal{S}_*^{(k_p)}$ satisfies

$$\lim_{p \rightarrow \infty} \mathcal{R}_{i^{(k_p)}, v_i^{(k_p)}}^{(k_p)} = 1.$$

For the rest of the argument, for notational convenience, we will avoid the dependence on the auction k_p . Note that for any value $v_i \in \mathcal{W}$, any bid in the interval $(\bar{\psi}_*, \infty)$ gives the agent strictly negative utility. Thus, the equilibrium utility for agent i at value v_i is

$$u_i(v_i, \mathcal{E}_{i,v_i}) \leq H x_i\left(\underline{\psi}_*\right) \mathcal{R}_{i,v_i} + H(1 - \mathcal{R}_{i,v_i}).$$

By assumption, the equilibrium strategy $\mathcal{E}_{i,v_i}$ is an ϵ -approximate equilibrium. Thus, the optimal utility at value v_i , which we denote by $u_i(v_i)$, satisfies

$$u_i(v_i) \leq H x_i\left(\underline{\psi}_*\right) \mathcal{R}_{i,v_i} + H(1 - \mathcal{R}_{i,v_i}) + \epsilon.$$

Thus, the allocation probability of agent i satisfies

$$x_i(b) \leq \frac{H x_i\left(\underline{\psi}_*\right) \mathcal{R}_{i,v_i} + H(1 - \mathcal{R}_{i,v_i}) + \epsilon}{v_i - b} \quad \text{for every bid } b \in \left[\underline{\psi}_*, \bar{\psi}_* - \epsilon^*\right]. \quad (\text{D.26})$$

We now establish a bound on the integral of the distribution function of $\mathcal{B}^{(k)}$ in the interval

$$\begin{aligned}
 & \left[\underline{\psi}_*, \bar{\psi}_* \right]: \\
 & \underbrace{\int_{\underline{\psi}_*}^{\bar{\psi}_*} \mathcal{B}^{(k)}(b) db}_{\mathcal{C}_i} \leq \int_{\underline{\psi}_*}^{\bar{\psi}_*} x_i(b) db = \int_{\underline{\psi}_*}^{\bar{\psi}_* - \epsilon^*} x_i(b) db + \int_{\bar{\psi}_* - \epsilon^*}^{\bar{\psi}_*} x_i(b) db, \\
 & \leq \int_{\underline{\psi}_*}^{\bar{\psi}_* - \epsilon^*} \frac{H x_i \left(\underline{\psi}_* \right) \mathcal{R}_{i, v_i} + H (1 - \mathcal{R}_{i, v_i}) + \epsilon}{v_i - b} db + \epsilon^*, \\
 & \leq \left(H x_i \left(\underline{\psi}_* \right) \mathcal{R}_{i, v_i} + H (1 - \mathcal{R}_{i, v_i}) + \epsilon \right) \log \left(\frac{v_i - \underline{\psi}_*}{v_i - \bar{\psi}_* + \epsilon^*} \right) + \epsilon^*, \\
 & \leq \left(H x_i \left(\underline{\psi}_* \right) \mathcal{R}_{i, v_i} + H (1 - \mathcal{R}_{i, v_i}) + \epsilon \right) \log \left(\frac{4H}{3\epsilon^*} \right) + \epsilon^*.
 \end{aligned}$$

Here, the first inequality follows from [Fact 6](#), the second inequality from [\(D.26\)](#) and the final inequality from the fact that $v_i \in [\bar{\psi}_* - \epsilon^*/4, \bar{\psi}_* - \epsilon^*/8]$. We will overload notation and denote a subsequence along which the integral $\mathcal{C}_i^{(k_p)}$ converges by $(k_p)_{p=1}^\infty$. By [Lemma 16](#) and the observation that $\lim_{p \rightarrow \infty} \mathcal{R}_{i^{(k_p)}, v_{i^{(k_p)}}}^{(k_p)} = 1$, we have

$$\lim_{p \rightarrow \infty} \mathcal{C}_i^{(k_p)} \leq \epsilon^* \log \left(\frac{4H}{3\epsilon^*} \right) + \epsilon^*.$$

Applying [Proposition 14](#), we have that for every sequence of auctions, there exists a subsequence of auctions $(\mathcal{A}^{(k_p)})_{p=1}^\infty$ such that

$$\lim_{p \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k_p)}, \mathcal{A}^{(k_p)})}{\mathcal{O}(\mathcal{A}^{(k_p)})} \geq 1 - \frac{2\epsilon^*}{\zeta} - \frac{1}{\zeta} \cdot \epsilon^* \log \left(\frac{4H}{3\epsilon^*} \right). \quad (\text{D.27})$$

Note that the bounds established in [\(D.24\)](#) and [\(D.27\)](#) are independent of the initial subsequence chosen. A straightforward calculation then establishes that along the sequence of

auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$, we have

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq \underbrace{1 - \zeta' \epsilon \left(1 + \log \left(\frac{H}{\epsilon}\right)\right)}_{\mathcal{L}}, \quad (\text{D.28})$$

where $\zeta' = (2 + \log(\frac{4}{3})) \cdot 1/\zeta$. To provide a brief argument, suppose the limit infimum is strictly smaller than $\mathcal{L}$. Then, there exists a subsequence of auctions such that the limit of the ratio of welfare to optimal welfare along this subsequence is strictly smaller than $\mathcal{L}$. This is a contradiction to what we establish in (D.27): for every sequence of auctions, there exists a subsequence of auctions along which the limit of the ratio is at least $\mathcal{L}$. The statement of the result is proved. $\square$

Lemma 19. *Consider any sequence of k -unit first-price auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$ and a corresponding sequence of $\epsilon^{(k)}$ -approximate Bayes-Nash equilibrium $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$ satisfying [Assumption 10](#). Let $\epsilon := \lim_{k \rightarrow \infty} \epsilon^{(k)}$ and suppose $\epsilon = 0$. Then,*

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq 1.$$

Proof. It suffices to establish that for every sequence of auctions and every $c > 0$, there exists extremizing sequence of bid pairs $\left\{(\underline{b}^{(k)}, \bar{b}^{(k)})\right\}_{k=1}^{\infty}$ and a corresponding subsequence of auctions $(\mathcal{A}^{(k_l)})_{l=1}^{\infty}$ along which

$$\lim_{l \rightarrow \infty} (\bar{b}^{(k_l)} - \underline{b}^{(k_l)}) < c.$$

Before we prove the statement above, we show why this implies the statement of the result. A direct application of [Proposition 14](#) implies

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq 1.$$

We now focus on the proof of the statement.

Suppose this is not the case. That is, there exists a constant $c^* > 0$ such that for any

extremizing sequence of bid pairs and any corresponding subsequence of auctions along which the bid difference converges, we have $\lim_{l \rightarrow \infty} (\bar{b}^{(k_l)} - \underline{b}^{(k_l)}) \geq c^*$. By Lemma 17, there exists a subsequence denoted by $(\mathcal{A}^{(k_m)})_{m=1}^\infty$ such that for every m large enough, there is a compact interval $\mathcal{W}^{(k_m)} \subset (\underline{\psi}_*^{(k_m)}, \bar{\psi}_*^{(k_m)})$ and a constant $\bar{u} > 0$ such that

- (a) $\text{len}(\mathcal{W}^{(k_m)}) \geq c/8$, and,
- (b) for every agent $i \in \mathcal{N}^{(k_m)}$, every value $v \in \mathcal{W}^{(k_m)}$ yields optimal utility at least $\bar{u}$.

For m large enough, every agent with a value in this interval obtains equilibrium utility arbitrarily close to their optimal utility which, by part (b), is at least $\bar{u}$. Here, $\bar{u}$ is positive. By Lemma 16, every bid in the interval $[0, \underline{\psi}_*^{(k_m)}]$ is allocated with vanishing probability.

We will use objects and results established in the proof of Lemma 18. Recall, $\mathcal{R}_*^{(k_m)}$ (see (D.25)) is the maximum measure of equilibrium bids an agent and a particular value in $\mathcal{W}^{(k_m)}$ has in the set $[0, \underline{\psi}_*^{(k_m)}] \cup (\bar{\psi}_*^{(k_m)}, \infty)$. Furthermore, in Lemma 18, we establish that any subsequence along which the limit of $\mathcal{R}_*^{(k_m)}$ exists satisfies $\lim_{p \rightarrow \infty} \mathcal{R}_*^{(k_p)} = 1$. Here, we index the subsequence with $(k_p)_{p=1}^\infty$.

Thus, there is a sequence of agent-value pairs $(\{i^{(k_p)}, v_{i^{(k_p)}}\})_{p=1}^\infty$ with the following properties:

- (a) for every $p \in \mathbb{N}$, the value $v_{i^{(k_p)}} \in \mathcal{W}^{(k_p)}$, and,
- (b) the limit of the measure of equilibrium bids in $\mathcal{S}_*^{(k_p)}$ satisfies

$$\lim_{p \rightarrow \infty} \mathcal{R}_{i^{(k_p)}, v_{i^{(k_p)}}}^{(k_p)} = 1.$$

For the rest of the argument, for notational convenience, we will avoid the dependence on the auction k_p . Note that for any value $v_i \in \mathcal{W}$, any bid in the interval $(\bar{\psi}_*, \infty)$ gives the agent strictly negative utility. Thus, the equilibrium utility for agent i at value v_i is

$$u_i(v_i, \mathcal{E}_{i, v_i}) \leq H x_i(\underline{\psi}_*) \mathcal{R}_{i, v_i} + H(1 - \mathcal{R}_{i, v_i}).$$

By Lemma 16, the allocation probability for any bid in the interval $[0, \underline{\psi}_*^{(k)}]$ converges to

zero as $p \rightarrow \infty$. Combining this with property (b) above, we have that agent i with value v gets vanishing equilibrium utility as $p \rightarrow \infty$. But, this is a contradiction to Part (b) in [Lemma 17](#) which states that every value in $\mathcal{W}$ receive equilibrium utility that is arbitrarily close to a positive $\bar{u}$ as $p \rightarrow \infty$. Thus, for every $c > 0$, there exists an extremizing sequence of bid pairs $\left\{ \left(\underline{b}^{(k)}, \bar{b}^{(k)} \right) \right\}_{k=1}^{\infty}$ and a corresponding subsequence of auctions $\left(\mathcal{A}^{(k_l)} \right)_{l=1}^{\infty}$ along which $\lim_{l \rightarrow \infty} \left(\bar{b}^{(k_l)} - \underline{b}^{(k_l)} \right) < c$. $\square$

Proof of [Theorem 7](#) Part (a). Consider any sequence of k -unit first-price auctions $\left(\mathcal{A}^{(k)} \right)_{k \in \mathbb{N}}$ and a corresponding sequence of Bayes-Nash equilibria $\left(\mathcal{E}^{(k)} \right)_{k \in \mathbb{N}}$. Fix a $k \in \mathbb{N}$. By [Assumption 5](#) and definition of $\bar{\psi}_*^{(k)}$ (see (4.1)), it must be that $\bar{\psi}_*^{(k)} < H$. Here, recall, H is an upper bound on the support of agent values.

Choose a $\omega^{(k)} > 0$ such that $\bar{\psi}_*^{(k)} + \omega^{(k)} < H$. By [Theorem 8](#), there exists a $\omega^{(k)}$ -approximate equilibrium $\tilde{\mathcal{E}}^{(k)}$ with the property that for every agent $i \in \mathcal{N}^{(k)}$, the bid distribution has no atoms. Let $\tilde{\psi}_*^{(k)}$ be the solution to (4.1) under profile $\tilde{\mathcal{E}}^{(k)}$. That is,

$$\tilde{\psi}_*^{(k)} := \min \left\{ b \geq 0 \mid \mathbb{E}[\tilde{\mathcal{T}}^{(k)}(b)] \leq k - k^{0.5+\delta} - 1 \right\}. \quad (\text{D.29})$$

Here, $\tilde{\mathcal{T}}^{(k)}(b) = \sum_{i \in \mathcal{N}^{(k)}} \tilde{\mathcal{T}}_i(b)$ where $\tilde{\mathcal{T}}_i(b)$ is an indicator random variable that takes value 1 if the bid $\tilde{\mathcal{E}}_i$ is strictly above b . We will establish that $\tilde{\psi}_*^{(k)} < H$. Recall, $\bar{\psi}_*^{(k)}$ is the solution to

$$\bar{\psi}_*^{(k)} := \min \left\{ b \geq 0 \mid \mathbb{E}[\mathcal{T}^{(k)}(b)] \leq k - k^{0.5+\delta} - 1 \right\}. \quad (\text{D.30})$$

Here, $\mathcal{T}^{(k)}(b) = \sum_{i \in \mathcal{N}^{(k)}} \mathcal{T}_i(b)$ where $\mathcal{T}_i(b)$ is an indicator random variable that takes value 1 if the bid $\mathcal{E}_i$ is strictly above b . We will show that

$$\mathbb{E} \left[\tilde{\mathcal{T}} \left(\bar{\psi}_*^{(k)} + \omega^{(k)} \right) \right] \leq k - k^{0.5+\delta} - 1. \quad (\text{D.31})$$

For the next part of the proof, we will use notation from the construction of $\tilde{\mathcal{E}}^{(k)}$ in [Section 4.5](#). Suppose $\bar{\psi}_*^{(k)}$ is not an atom. By construction, any bid that is strictly above $\bar{\psi}_*^{(k)}$ is mapped

almost surely to a bid strictly above

$$\underbrace{\bar{\psi}_*^{(k)} + \sum_{\{p \in \mathcal{P}: p < \bar{\psi}_*^{(k)}\}} s_p}_{\tilde{s}}.$$

See proof of [Theorem 8](#) Part (c) for a detailed argument. Furthermore, any bid that is strictly below $\bar{\psi}_*^{(k)}$ is mapped almost surely to a bid smaller than $\tilde{s}$. Since $\bar{\psi}_*^{(k)}$ is not an atom by assumption, the expected total bids above $\tilde{s}$ in the bid profile $\tilde{\mathcal{E}}^{(k)}$ is at most $k - k^{0.5+\delta} - 1$. Since $\sum_{\{p \in \mathcal{P}: p < \bar{\psi}_*^{(k)}\}} s_p < \omega^{(k)}$, the expected total equilibrium bids strictly above $\bar{\psi}_*^{(k)} + \omega^{(k)}$ in profile $\tilde{\mathcal{E}}$ is at most $k - k^{0.5+\delta} - 1$.

An identical argument establishes that when $\bar{\psi}_*^{(k)}$ is an atom, the total expected equilibrium bids strictly above $\bar{\psi}_*^{(k)} + \sum_{\{p \in \mathcal{P}: p \leq \bar{\psi}_*^{(k)}\}} s_p$ is at most $k - k^{0.5+\delta} - 1$. Since $\sum_{\{p \in \mathcal{P}: p \leq \bar{\psi}_*^{(k)}\}} s_p < \omega^{(k)}$, the expected total equilibrium bids strictly above $\bar{\psi}_*^{(k)} + \omega^{(k)}$ in bid profile $\tilde{\mathcal{E}}$ is at most $k - k^{0.5+\delta} - 1$. By definition of $\widetilde{\psi}_*^{(k)}$, we have

$$\widetilde{\psi}_*^{(k)} < \bar{\psi}_*^{(k)} + \omega^{(k)} < H.$$

Thus, there exists a sequence of perturbation parameters $(\omega^{(k)})_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \omega^{(k)} = 0,$$

and the constructed bid profiles $\tilde{\mathcal{E}}^{(k)}$ satisfy $\widetilde{\psi}_*^{(k)} < H$ for every $k \in \mathbb{N}$. Applying [Lemma 19](#), we have

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\tilde{\mathcal{E}}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} = 1. \quad (\text{D.32})$$

By Part (c) of [Theorem 8](#), the welfare of equilibrium $\tilde{\mathcal{E}}^{(k)}$ and $\mathcal{E}^{(k)}$ are identical. Substituting

in (D.32), we get

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} = 1.$$

This proves the statement of the result. $\square$

Proof of Theorem 7 Part (b). Consider any sequence of k -unit first-price auctions $(\mathcal{A}^{(k)})_{k \in \mathbb{N}}$ and a corresponding sequence of $\epsilon^{(k)}$ -approximate Bayes-Nash equilibria $(\mathcal{E}^{(k)})_{k \in \mathbb{N}}$. By Assumption 5, the threshold bid $\bar{\psi}_*^{(k)} < H$ for every $k \in \mathbb{N}$. As established in the proof of Part (a) of Theorem 7 and by Theorem 8, we can pick a sequence of perturbations $(\omega^{(k)})_{k \in \mathbb{N}}$ and a corresponding sequence of constructed equilibria $(\tilde{\mathcal{E}}^{(k)})_{k \in \mathbb{N}}$ satisfying the following:

- (i) $\lim_{k \rightarrow \infty} \omega^{(k)} = 0$.
- (ii) $\widetilde{\psi}_*^{(k)} < H$ for every $k \in \mathbb{N}$
- (iii) $\tilde{\mathcal{E}}^{(k)}$ is a $(\epsilon^{(k)} + \omega^{(k)})$ -approximate equilibrium having no atoms.

Applying Lemma 18, we have

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\tilde{\mathcal{E}}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq 1 - \zeta' \epsilon^* \left(1 + \log \left(\frac{H}{\epsilon^*} \right) \right). \quad (\text{D.33})$$

By Theorem 8 Part (c), the welfare of equilibrium $\tilde{\mathcal{E}}^{(k)}$ and $\mathcal{E}^{(k)}$ are identical. Substituting in (D.33), we get

$$\liminf_{k \rightarrow \infty} \frac{\mathcal{W}(\mathcal{E}^{(k)}, \mathcal{A}^{(k)})}{\mathcal{O}(\mathcal{A}^{(k)})} \geq 1 - \zeta' \epsilon^* \left(1 + \log \left(\frac{H}{\epsilon^*} \right) \right).$$

The statement of the result is proved. $\square$

D.1.6 Proof of Proposition 15

Consider the sequence of distribution functions $(\mathcal{B}^{(k)}(\cdot))_{k \in \mathbb{N}}$. By Helly's [1912] selection theorem, the sequence admits a pointwise convergent subsequence. We want to establish

that along any such subsequence, the limit denoted by $\mathcal{B}^*(\cdot)$ satisfies

$$\mathcal{B}^*(b) = \begin{cases} 1 & \text{if } b \geq \psi^* \\ 0 & \text{if } b < \psi^* \end{cases}. \quad (\text{D.34})$$

Consider any pointwise convergent subsequence that does not satisfy (D.34). We will overload notation and denote the elements of this subsequence by the index $k \in \mathbb{N}$. By assumption, there exist two bids $\underline{b}^*$ and $\bar{b}^*$ such that:

$$(a) \quad 0 < \mathcal{B}^*(\underline{b}^*) \leq \mathcal{B}^*(\bar{b}^*) < 1, \text{ and,}$$

$$(b) \quad \bar{b}^* - \underline{b}^* > 0.$$

As established in the proof of Lemma 19, for every sequence of auctions and every $c > 0$, there exists extremizing sequence of bid pairs $\left\{ \left(\underline{b}^{(k)}, \bar{b}^{(k)} \right) \right\}_{k=1}^{\infty}$ and a corresponding subsequence of auctions $(\mathcal{A}^{(k_l)})_{l=1}^{\infty}$ along which

$$\lim_{l \rightarrow \infty} \left(\bar{b}^{(k_l)} - \underline{b}^{(k_l)} \right) < c.$$

We will now establish a contradiction. Since $\mathcal{B}^*(\cdot)$ is monotonic and the distribution function $\mathcal{B}^{(k)}(\cdot)$ pointwise converges to $\mathcal{B}^*(\cdot)$, by $0 < \mathcal{B}^*(\underline{b}^*) \leq \mathcal{B}^*(\bar{b}^*) < 1$ and $\bar{b}^* - \underline{b}^* > 0$, there cannot exist an extremizing sequence of bid pairs such that the bid difference is at most $\bar{b}^* - \underline{b}^*$. This is a contradiction to the result established in the paragraph above. The statement of the result is proved.

D.1.7 Proof of Theorem 8

We begin by proving Part (b), then prove Part (c) and finish by proving Part (a).

Proof of Part (b). This follows by construction of bid distributions $\tilde{\mathcal{E}}$. □

Proof of Part (c). We will compare the allocation probability $x_i(\cdot)$ under the bid profile $\mathcal{E}$ with the allocation probability $\tilde{x}_i(\cdot)$ under the bid profile $\tilde{\mathcal{E}}$. Consider any agent i and any

bid b_i . In particular, we will establish the following result.

Lemma 20. *Consider any agent $i \in \mathcal{N}$ and bid b_i . The allocation probability $x_i(\cdot)$ and $\tilde{x}_i(\cdot)$ satisfy the following:*

- (i) *if $b_i \in \mathcal{P}$, then for the bid $\tilde{b}_i$ defined in (4.3) we have $x_i(b_i) = \tilde{x}_i(\tilde{b}_i)$.*
- (ii) *if $b_i \notin \mathcal{P}$, then for the random bid $\tilde{b}_i$ defined in (4.4) we have $x_i(b_i) = \mathbb{E} \left[\tilde{x}_i(\tilde{b}_i) \right]$.*

Before we prove the lemma, we demonstrate how it implies Part (c) of Theorem 8, that is,

$$welfare(\mathcal{E}) = welfare(\tilde{\mathcal{E}}).$$

Recall, we denote the equilibrium bid distribution of agent i by $\mathcal{E}_{i,v}$. The welfare under equilibrium $\mathcal{E}$ is

$$welfare(\mathcal{E}) = \mathbb{E}_{\mathbf{v} \sim \mathcal{V}} \left[\sum_{i \in \mathcal{N}} v_i \mathbb{E}_{b_i \sim \mathcal{E}_{i,v_i}} [x_i(b_i)] \right]. \quad (\text{D.35})$$

By construction of $\tilde{\mathcal{E}}$ and Lemma 20, we have

$$\mathbb{E}_{b_i \sim \mathcal{E}_{i,v_i}} [x_i(b_i)] = \mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[\tilde{x}_i(\tilde{b}_i) \right].$$

Substituting in (D.35), we have

$$welfare(\mathcal{E}) = \mathbb{E}_{\mathbf{v} \sim \mathcal{V}} \left[\sum_{i \in \mathcal{N}} v_i \mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[\tilde{x}_i(\tilde{b}_i) \right] \right] = welfare(\tilde{\mathcal{E}}).$$

The rest of the proof focuses on providing arguments to establish Lemma 20. We begin by proving Part (i) and then prove Part (ii).

Proof of Part (i). Suppose $b_i \notin \mathcal{P}$. We will establish that $x_i(b_i) = \tilde{x}_i(\tilde{b}_i)$ where $\tilde{b}_i$ is defined in (4.3). To show this, it suffices to prove that the order of bids between bid profiles $\mathbf{b}$ and

$\tilde{b}$ given by (4.3) and (4.4), is preserved as follows: for any agent j and bid b_j ,

$$\begin{aligned} & \text{if } b_i > b_j, \text{ then } \tilde{b}_i > \tilde{b}_j \text{ almost surely and} \\ & \text{if } b_i < b_j, \text{ then } \tilde{b}_i < \tilde{b}_j \text{ almost surely.} \end{aligned} \tag{D.36}$$

. We begin by showing that if $b_j < b_i$, the sampled bid $\tilde{b}_j < \tilde{b}_i$ almost surely. This is because of the following:

- if $b_j \notin \mathcal{P}$, then because $b_j < b_i$, we have

$$\tilde{b}_j = b_j + \sum_{\{p \in \mathcal{P}: p < b_j\}} s_p < b_i + \sum_{\{p \in \mathcal{P}: p < b_i\}} s_p = \tilde{b}_i.$$

- if $b_j \in \mathcal{P}$, by definition (4.4), the sampled bid $\tilde{b}_j$ is at most $b_j + \sum_{\{p \in \mathcal{P}: p \leq b_j\}} s_p$, almost surely. Because $b_j < b_i$, we have

$$b_j + \sum_{\{p \in \mathcal{P}: p \leq b_j\}} s_p < b_i + \sum_{\{p \in \mathcal{P}: p < b_i\}} s_p = \tilde{b}_i.$$

It follows that $\tilde{b}_j < \tilde{b}_i$ almost surely.

Next, for any agent j and bid b_j satisfying $b_j > b_i$, the sampled bids satisfy $\tilde{b}_j > \tilde{b}_i$ almost surely. This is because of the following:

- if $b_j \notin \mathcal{P}$, then because $b_j > b_i$, we have

$$\tilde{b}_j = b_j + \sum_{\{p \in \mathcal{P}: p < b_j\}} s_p > b_i + \sum_{\{p \in \mathcal{P}: p < b_i\}} s_p = \tilde{b}_i.$$

- if $b_j \in \mathcal{P}$, by definition (4.4), the sampled bid $\tilde{b}_j$ is at least $b_j + \sum_{\{p \in \mathcal{P}: p < b_j\}} s_p$, almost surely. Because $b_j > b_i$, we have

$$b_j + \sum_{\{p \in \mathcal{P}: p < b_j\}} s_p > b_i + \sum_{\{p \in \mathcal{P}: p < b_i\}} s_p = \tilde{b}_i.$$

It follows $\tilde{b}_j > \tilde{b}_i$ almost surely.

Finally, the bid b_i is not an atom. Thus, for any agent j and bid b_j , the probability that $b_i = b_j$ is zero. Combining this with the observation that the order of bids between profiles $\mathbf{b}$ and $\tilde{\mathbf{b}}$ is preserved as defined in (D.36), we have that the allocation probability $x_i(b_i) = \tilde{x}_i(\tilde{b}_i)$.

Proof of Part (ii). Suppose $b_i \in \mathcal{P}$. Here, the mapped bid $\tilde{b}_i$ is sampled according to

$$\tilde{b}_i \sim b_i + \sum_{\{p \in \mathcal{P}: p < b_i\}} s_p + s_b \cdot U[0, 1].$$

We will establish that $x_i(b_i) = \mathbb{E} \left[\tilde{x}_i(\tilde{b}_i) \right]$. Consider any profile of bids $\mathbf{b}$. To show this, it suffices to establish the following:

- any bid b_j that does not tie with b_i preserves the ordering as defined in (D.36), and,
- for any set of bids b_j that tie with b_i , the mapped random bids $\tilde{b}_j$ and $\tilde{b}_i$ simulate uniform tie-breaking.

An argument identical to that presented in Case 1 establishes Part (i). To show Part (ii), let $\mathcal{J}^*$ be the set of agents that have an atom at b_i such that the sampled bid $b_j = b_i$. By assumption that the auction breaks ties uniform randomly, under bid profile $\mathbf{b}$, agent i 's allocation probability satisfies

$$x_i(\mathbf{b}) = \min \left\{ \frac{\max \left\{ k - \sum_{j \in \mathcal{N}} \mathbf{1}[b_j > b_i], 0 \right\}}{|\mathcal{J}^*|}, 1 \right\}, \quad (\text{D.37})$$

where $\mathbf{1}[\cdot]$ is the indicator function. Under the mapped profile $\tilde{\mathbf{b}}$ and by Part (i), only agents in $\mathcal{J}^*$ compete for the remaining $\max \left\{ k - \sum_{j \in \mathcal{N}} \mathbf{1}[b_j > b_i], 0 \right\}$ units. Furthermore, by (4.4), for every $j \in \mathcal{J}^*$, we have

$$\tilde{b}_j \sim b_i + \sum_{p \in \mathcal{P}: p < b_i} s_p + s_{b_i} \cdot U[0, 1],$$

where the randomness is independent across agents. Observe that the mapped bid $\tilde{b}_j$ is

independent and identical across all agents in $\mathcal{J}^*$. Thus, under the random profile $\tilde{\mathbf{b}}$, the expected allocation probability for agent i is

$$\mathbb{E} [\tilde{x}_i(\tilde{\mathbf{b}})] = \min \left\{ \frac{\max \left\{ k - \sum_{j \in \mathcal{N}} \mathbf{1}[b_j > b_i], 0 \right\}}{|\mathcal{J}^*|}, 1 \right\} = x_i(\mathbf{b}), \quad (\text{D.38})$$

where the second equality follows from (D.37). Taking an expectation over all bid profiles $\mathbf{b}$ gives the result $x_i(b_i) = \mathbb{E} [\tilde{x}_i(\tilde{b}_i)]$. Part (c) of the theorem follows. $\square$

Proof of Part (a). By definition of an ϵ -approximate equilibrium, for every agent $i \in \mathcal{N}$ and every value v_i , the agent's expected utility is

$$\mathbb{E}_{b_i \sim \mathcal{E}_{i,v_i}} [(v_i - b_i) x_i(b_i)] \geq \sup_{\bar{b} \geq 0} (v_i - \bar{b}) x_i(\bar{b}) - \epsilon. \quad (\text{D.39})$$

We will now establish a bound on the utility under equilibrium $\tilde{\mathcal{E}}$. For any agent i and value v_i , we show the expected utility satisfies

$$\mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[(v_i - \tilde{b}_i) \tilde{x}_i(\tilde{b}_i) \right] \geq \mathbb{E}_{b_i \sim \mathcal{E}_{i,v_i}} [(v_i - b_i) x_i(b_i)] - \delta. \quad (\text{D.40})$$

By construction of $\tilde{\mathcal{E}}$, the bid $\tilde{b}_i$ is mapped from a bid b_i that is either an atom or not (see (4.4) and (4.3)). In both cases, we have that the difference $\tilde{b}_i - b_i \in (0, \omega)$ almost surely. Let $\tilde{\mathcal{P}}$ denote the event that bid $\tilde{b}_i$ is mapped from an atom and $\tilde{\mathcal{P}}^c$ denote its complement. We can write agent i 's expected utility as

$$\begin{aligned} \mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[(v_i - \tilde{b}_i) \tilde{x}_i(\tilde{b}_i) \right] &= \mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[(v_i - \tilde{b}_i) \tilde{x}_i(\tilde{b}_i) \mid \tilde{\mathcal{P}}^c \right] \mathbb{P} [\tilde{\mathcal{P}}^c] \\ &\quad + \mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[(v_i - \tilde{b}_i) \tilde{x}_i(\tilde{b}_i) \mid \tilde{\mathcal{P}} \right] \mathbb{P} [\tilde{\mathcal{P}}]. \end{aligned} \quad (\text{D.41})$$

We will rewrite the right-hand side in terms of equilibrium $\mathcal{E}$. Consider the first term. By Part (i) of Lemma 20 and using $\tilde{b}_i - b_i < \omega$, we have

$$\mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[(v_i - \tilde{b}_i) \tilde{x}_i(\tilde{b}_i) \mid \tilde{\mathcal{P}}^c \right] \mathbb{P} [\tilde{\mathcal{P}}^c] \geq (\mathbb{E}_{b_i \sim \mathcal{E}_{i,v_i}} [(v_i - b_i) x_i(b_i) \mid b_i \notin \mathcal{P}] - \omega) \mathbb{P} [b_i \notin \mathcal{P}], \quad (\text{D.42})$$

where, recall, $\mathcal{P}$ is the set of atoms. For the second term, by Part (ii) of [Lemma 20](#) and using $\tilde{b}_i - b_i < \omega$ almost surely, we have

$$\mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[\left(v_i - \tilde{b}_i \right) \tilde{x}_i \left(\tilde{b}_i \right) | \tilde{\mathcal{P}} \right] \mathbb{P} \left[\tilde{\mathcal{P}} \right] \geq \left(\mathbb{E}_{b_i \sim \mathcal{E}_{i,v_i}} \left[\left(v_i - b_i \right) x_i(b_i) | b_i \in \mathcal{P} \right] - \omega \right) \mathbb{P} [b_i \in \mathcal{P}]. \quad (\text{D.43})$$

Substituting [\(D.42\)](#) and [\(D.43\)](#) in [\(D.41\)](#), we get the desired inequality in [\(D.40\)](#).

To establish that $\tilde{\mathcal{E}}$ is an $(\epsilon + \omega)$ -approximate equilibrium, it is sufficient to show

$$\sup_{\bar{b} \geq 0} (v_i - \bar{b}) x_i(\bar{b}) \geq \max_{\bar{b} \geq 0} (v_i - \bar{b}) \tilde{x}_i(\bar{b}). \quad (\text{D.44})$$

The right-hand side in [\(D.44\)](#) is well defined. To see why, bid distributions in $\tilde{\mathcal{E}}$ have no atoms and thus $\tilde{x}_i(\cdot)$ is continuous. Furthermore, every bid $\bar{b}$ in the compact interval $[0, v_i]$ obtains strictly greater utility than bids strictly greater than v_i . Since a continuous function over a compact interval achieves its maximum, the right-hand side is well defined. Now, our approach is to show that for any solution to the right-hand side, there is a corresponding bid which gives the agent weakly greater utility under allocation $x_i(\cdot)$.

Let $\tilde{b}$ denote an optimal solution to the right-hand side of [\(D.44\)](#). Suppose $\tilde{b}$ is mapped from a bid b that is not an atom (see [\(4.3\)](#)). Then by Part (i) of [Lemma 20](#), we have $x_i(b) = \tilde{x}_i(\tilde{b})$. Furthermore, by construction in [\(4.3\)](#), we have $\tilde{b} > b$. Thus,

$$\sup_{\bar{b} \geq 0} (v_i - \bar{b}) x_i(\bar{b}) \geq (v_i - b) x_i(b) > (v_i - \tilde{b}) \tilde{x}_i(\tilde{b}).$$

Now suppose, $\tilde{b}$ is mapped from a bid b that is an atom (see [\(4.4\)](#)). Since $\tilde{b} > b$ and the number of atoms are countable, there exists a bid b^* in the interval $(b, \tilde{b})$ such that b^* is not an atom. By definition of the map in [\(4.3\)](#), we have

$$\tilde{b}^* = b^* + \sum_{p \in \mathcal{P}: p < b^*} s_p > b + \sum_{p \in \mathcal{P}: p \leq b} s_p \geq \tilde{b}.$$

By monotonicity of the allocation rule, we have $\tilde{x}_i(\tilde{b}^*) \geq \tilde{x}_i(\tilde{b})$. But by Part (i) of [Lemma 20](#),

we have $x_i(b^*) = \tilde{x}_i(\tilde{b}^*)$. Finally, using $b^* < \tilde{b}$, we have

$$\sup_{\bar{b} \geq 0} (v_i - \bar{b}) x_i(\bar{b}) \geq (v_i - b^*) x_i(b^*) > (v_i - \tilde{b}) \tilde{x}_i(\tilde{b}).$$

This establishes (D.44).

Finally, consider the following chain of inequalities:

$$\begin{aligned} \mathbb{E}_{\tilde{b}_i \sim \tilde{\mathcal{E}}_{i,v_i}} \left[(v_i - \tilde{b}_i) \tilde{x}_i(\tilde{b}_i) \right] &\geq \mathbb{E}_{b_i \sim \mathcal{E}_{i,v_i}} [(v_i - b_i) x_i(b_i)] - \omega \quad (\text{from (D.40)}), \\ &\geq \sup_{\bar{b} \geq 0} (v_i - \bar{b}) x_i(\bar{b}) - (\epsilon + \omega) \quad (\text{from (D.39)}), \\ &\geq \max_{\bar{b} \geq 0} (v_i - \bar{b}) \tilde{x}_i(\bar{b}) - (\epsilon + \omega) \quad (\text{from (D.44)}). \end{aligned}$$

Since this holds for every agent i and every value v_i , the bid profile $\tilde{\mathcal{E}}$ is an $(\epsilon + \omega)$ -approximate equilibrium. $\square$

D.2 Continuity and Monotonicity of Equilibrium Bid Distributions

This section focuses on Bayes-Nash equilibrium. [Section D.2.1](#) derives some standard properties of equilibrium bid distributions in the multi-unit setting. [Section D.2.2](#) applies a neighborhood deviation argument to show that this bid distribution is continuous and strictly increasing.

D.2.1 Properties of Equilibrium Bid Distributions

Before proving our main result, we introduce key definitions used throughout this section. We define γ_i as the infimum and λ_i as the supremum of the support of agent i 's equilibrium bid distribution:

$$\gamma_i := \inf(\text{supp}(\mathcal{E}_i)) \text{ and } \lambda_i := \sup(\text{supp}(\mathcal{E}_i)).$$

Additionally, we define $\mathbf{\Gamma}$ as the collection of γ_i values, sorted in descending order:

$$\mathbf{\Gamma} := \{\gamma_i\}_{i \in \mathcal{N}}.$$

We denote the i^{th} element in this ordered set by $\gamma_{(i)}$. We define the infimum of the k -th highest bid $\mathcal{B}^{(k)}$ as:

$$\gamma^{(k)} := \inf(\text{supp}(\mathcal{B}^{(k)})).$$

Lemma 21. *The infimum of the support of the bid distribution of the k^{th} -highest agent is the k^{th} element in the order $\mathbf{\Gamma}$, that is,*

$$\gamma^{(k)} = \gamma_{(k)}.$$

Proof. Consider any agent i such that γ_i is in the top k elements of the order $\mathbf{\Gamma}$. By definition of γ_i , this agent almost surely has a bid $b \geq \gamma_i$. Since this holds for any agent corresponding to a top k element in the order $\mathbf{\Gamma}$, the bid of the k^{th} highest agent is almost surely at least $\gamma_{(k)}$. Thus it must be that

$$\gamma^{(k)} \geq \gamma_{(k)}.$$

We will now show that in fact $\gamma^{(k)} = \gamma_{(k)}$. Let us denote the agent with the k^{th} highest element in $\mathbf{\Gamma}$ by i^* , that is, $\gamma_{i^*} = \gamma_{(k)}$. Suppose $\gamma^{(k)} > \gamma_{(k)}$. By definition of γ_{i^*} , agent i^* bids in the interval $[\gamma_{(k)}, \gamma^{(k)})$ with strictly positive probability. By definition of $\gamma_{(k)}$, the bid of the k^{th} highest agent lies in the interval $[\gamma_{(k)}, \gamma^{(k)})$ with probability 0. This is a contradiction to agent i^* having their γ_{i^*} in the top k elements of $\mathbf{\Gamma}$. $\square$

The result above demonstrated an equivalence of the infimum of equilibrium bid distributions and the distribution of the k^{th} highest bid. In addition to this, we prove a crucial result which shows that the $k + 1$ highest suprema of equilibrium bid distributions must all be the same.

Lemma 22 (Support upper bound equivalence). *Suppose $\lambda_{(i)}$ is the i^{th} highest value in the*

set $\{\lambda_j\}_{j \in \mathcal{N}}$. Then,

$$\lambda_{(1)} = \dots = \lambda_{(k+1)}.$$

Proof. Suppose for a contradiction it is not true that $\lambda_{(1)} = \dots = \lambda_{(k+1)}$. Then, since $\lambda_{(1)} \geq \dots \geq \lambda_{(k+1)}$, we must have

$$\lambda_{(k+1)} < \lambda_{(1)}.$$

This implies the following: $\lambda_{(k+1)}$ is finite and the region $(\lambda_{(k+1)}, \lambda_{(1)}] \subseteq \mathbb{R}$ is non-empty. In addition to this, there must exist an agent i with $\lambda_i = \lambda_{(1)}$ who bids in $(\lambda_{(k+1)}, \lambda_{(1)}]$ with positive probability, and for all $z \in (\lambda_{(k+1)}, \lambda_{(1)}]$, this agent's interim allocation is $x_i(z) = 1$.

This permits agent i to adjust their bidding strategy such that, whenever they would have bid $b_i \in (\lambda_{(k+1)}, \lambda_{(1)}]$, they instead bid

$$b_i^* := \lambda_{(k+1)} + (b_i - \lambda_{(k+1)})/2.$$

This keeps their interim allocation $x_i(b_i^*) = x_i(b_i) = 1$ constant, but strictly decreases their bid, since $b_i^* < b_i$. Thus, this new strategy strictly increases agent i 's ex-ante utility, and so their original bidding strategy must not have been optimal, which is a contradiction. $\square$

We now present some standard results that have been established in the single-unit setting (see for example [Jin and Lu \[2023\]](#)) and show their equivalent results in the multi-unit setting. The first result concerns an agent's interim allocation rule and shows several standard properties. Specifically, an agent's interim allocation weakly increases in their bid. Additionally, in equilibrium, a bid above $\gamma^{(k)}$ secures a unit with strictly positive probability, while a bid below $\gamma^{(k)}$ has a zero probability in receiving a unit.

Lemma 23. *The interim allocation rule $x_i(b)$ for any agent i satisfies the following properties:*

(a) $x_i(b)$ is weakly increasing in b .

(b) $x_i(b)$ is bounded as

$$\sup\{\mathcal{B}_{-i}^{(k)}(t) : t < b\} \leq x_i(b) \leq \mathcal{B}_{-i}^{(k)}(b) \quad \text{for all } b.$$

(c) Consider any bid b in the support of the bid distribution $\mathcal{E}_i$. It has interim allocation

(i) $x_i(b) = 0$ when $b < \gamma^{(k)}$.

(ii) $x_i(b) > 0$ when $b > \gamma^{(k)}$.

Proof. The allocation rule of agent i for a bid b is

$$x_i(b) = \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b] + \mathbb{P}[\mathcal{B}_{-i}^{(k)} = b] \mathbb{P}[i \text{ wins in a tie-break at bid } b]. \quad (\text{D.45})$$

We will use this expression to prove the stated properties.

(a) *Proof of Part (a):* It is clear from (D.45) that the interim allocation is bounded as

$$x_i(b) \leq \mathbb{P}[\mathcal{B}_{-i}^{(k)} \leq b].$$

Consider an arbitrary $\epsilon > 0$ and a bid $b' = b + \epsilon$. The interim allocation for bid b' satisfies

$$\begin{aligned} x_i(b') &= \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b'] + \mathbb{P}[\mathcal{B}_{-i}^{(k)} = b'] \mathbb{P}[i \text{ wins in a tie-break at bid } b'], \\ &\geq \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b + \epsilon], \\ &\geq \mathbb{P}[\mathcal{B}_{-i}^{(k)} \leq b], \\ &\geq x_i(b). \end{aligned}$$

The equality follows from (D.45) and the first inequality follows from the fact that a probability is always non-negative. Since, we show that for any $\epsilon > 0$, the interim allocation $x_i(b + \epsilon) \geq x_i(b)$, the statement in Part (a) follows.

(b) *Proof of Part (b):* As was shown in the proof of Part (a), an upper bound on the interim

allocation is

$$x_i(b) \leq \mathcal{B}_{-i}^{(k)}(b).$$

To show the lower bound, it holds that

$$\mathcal{B}_{-i}^{(k)}(t) \leq \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b] \quad \text{for all } t < b.$$

It is straightforward to see that

$$\begin{aligned} \sup\{\mathcal{B}_{-i}^{(k)}(t) : t < b\} &\leq \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b], \\ &\leq x_i(b). \end{aligned}$$

The statement of Part (b) follows.

- (c) *Proof of Part (c):* Consider an agent i . We will now analyze its competing bid distribution $\mathcal{B}_{-i}^{(k)}$. Let us denote the infimum of its support by $\gamma_{-i}^{(k)}$. Moreover, let us define the collection

$$\mathbf{\Gamma}_{-i} := \{\gamma_j\}_{j \neq i}$$

An application of [Lemma 21](#) tells us that

$$\gamma_{-i}^{(k)} = \Gamma_{-i}^{(k)}. \tag{D.46}$$

Suppose agent i is such that $\gamma_i < \gamma^{(k)}$. The infimum of the support of the competing bid distribution satisfies

$$\begin{aligned} \gamma_{-i}^{(k)} &= \Gamma_{-i}^{(k)}, \\ &= \gamma^{(k)}, \\ &= \gamma^{(k)}. \end{aligned}$$

The first equality follows from (D.46), the second follows from the assumption $\gamma_i < \gamma^{(k)}$

and the third follows from [Lemma 21](#). Consider any bid b in the support of agent i 's bid distribution. In the case $b < \gamma^{(k)}$, the interim allocation $x_i(b) = 0$ because the bid lies strictly below the infimum of the support of the competing bid distribution. In the case $b > \gamma^{(k)}$, he is allocated with strictly positive probability because the bid lies strictly above the support of the competing bid distribution.

Suppose agent i is such that $\gamma_i \geq \gamma^{(k)}$. By definition of γ_i , the agent almost surely has a bid $b \geq \gamma_i$. Thus, Part (i) vacuously holds true. The infimum of the support of the competing bid distribution

$$\begin{aligned}\gamma_{-i}^{(k)} &= \Gamma_{-i}^{(k)}, \\ &\leq \gamma^{(k)}, \\ &= \gamma^{(k)}.\end{aligned}$$

The second equality follows from observing that removing an element from the order $\mathbf{\Gamma}$ only decreases the value of entry k . It follows that any bid $b > \gamma^{(k)}$ is allocated with strictly positive probability because the infimum of the competing bid distribution is strictly below the bid.

The 3 parts of the result statement are proved. □

Next, we examine the equilibrium strategy of an agent. The agent obtains positive utility if and only their value is strictly greater than the infimum $\gamma^{(k)}$. Furthermore, they submit a bid at or above this threshold if their value exceeds it strictly.

Lemma 24. *Consider an agent i and value $v \in \text{supp}(V_i)$. The equilibrium strategy $s_i(\cdot)$ for any agent i satisfies the following properties:*

- (a) *If $v \leq \gamma^{(k)}$, the optimal utility $\mathcal{U}_i(v) = 0$ and the equilibrium bid $s_i(v) \leq \gamma^{(k)}$ almost surely over the random strategy $s_i(\cdot)$.*
- (b) *If $v > \gamma^{(k)}$, the optimal utility $\mathcal{U}_i(v) > 0$ and the equilibrium bid $\gamma^{(k)} \leq s_i(v) < v$ almost surely over the random strategy $s_i(\cdot)$.*

Proof. We will consider two cases: one where $\gamma_i < \gamma^{(k)}$ and the other where $\gamma_i \geq \gamma^{(k)}$.

Suppose $\gamma_i < \gamma^{(k)}$. As we have shown in the proof of [Lemma 23](#), the infimum of the support of the competing bid distribution $\gamma_{-i}^{(k)}$ is equal to $\gamma^{(k)}$.

- (i) Proof of Part (a): Consider a value $v \leq \gamma^{(k)}$. Let it be the case that $v \leq \gamma^{(k)}$. If the agent were to bid $b > \gamma^{(k)}$, this would give them utility $\mathcal{U}_i(v, b) < 0$ because he is allocated with strictly positive probability. If the agent bid $b < \gamma^{(k)}$, the agent is allocated with probability 0 because the infimum of the support of the competing bid distribution is $\gamma^{(k)}$. It follows that they receive $\mathcal{U}_i(v, b) = 0$ for such a bid. If the agent bid $b = \gamma^{(k)}$, the utility he gets for such a bid is at most 0 (the utility would be 0 in the case he is not allocated an item). Thus, the optimal utility for such a value is $\mathcal{U}_i(v) = 0$. Moreover, the bids which guarantee this utility must be at most $\gamma^{(k)}$. Thus, the equilibrium bid $s_i(v) \leq \gamma^{(k)}$ almost surely.
- (ii) Proof of Part (b): Consider a value $v > \gamma^{(k)}$. We will show that the bid $b = \frac{v + \gamma^{(k)}}{2}$ gives the agent a strictly positive utility. Observe that the bid $b > \gamma^{(k)}$ and has allocation $x_i(b) > 0$ because the bid is strictly greater than the infimum of the support of the competing bid distribution $\gamma^{(k)}$. Since the bid $b < v$, the utility for this bid $\mathcal{U}_i(v, b) > 0$. The optimal bid can only do better. Thus, the optimal utility $\mathcal{U}_i(v) > 0$. We will now show that $\gamma^{(k)} \leq s_i(v) < v$ almost surely. If the agent bids $b < \gamma^{(k)}$, his allocation is $x_i(b) = 0$ (the bid is strictly smaller than the infimum of the support of the competing bid distribution). This would give the agent a utility of 0 and thus cannot be optimal. It follows that $\gamma^{(k)} \leq s_i(v) < v$ almost surely (the bound $s_i(v) < v$ follows from the observation that a bid $b \geq v$ will be allocated with strictly positive probability but given a utility $\mathcal{U}_i(v, b) \leq 0$, thus not being optimal).

Suppose $\gamma_i \geq \gamma^{(k)}$. We will state two observations which will be helpful in proving both parts of the statement. First, the infimum of the support of $\mathcal{E}_i$ is at least $\gamma^{(k)}$. Thus, the equilibrium bid of this agent is at least $\gamma^{(k)}$ almost surely. Second, as we have shown in the proof of [Lemma 23](#), the infimum of the support of the competing bid distribution $\gamma_{-i}^{(k)}$ is at most $\gamma^{(k)}$.

- (i) Proof of Part (a): Consider a value $v \leq \gamma^{(k)}$. We will show that it must be the case

equilibrium bids $s_i(v)$ give the agent 0 utility. Since the equilibrium strategy is such that $s_i(v) \geq \gamma^{(k)}$ almost surely, the agent must be getting at most 0 utility. It cannot be the case that an agent gets strictly negative utility at equilibrium. Thus, the optimal utility is 0. We will now analyze the equilibrium bid strategy $s_i(v)$. For this value v , it cannot be the case that a bid b in the support of $s_i(v)$ lies strictly above $\gamma^{(k)}$. The agent's utility conditional on such a bid is strictly negative. The agent can strictly increase utility by moving probability measure around b to a bid that gives him non-negative utility (such a bid must exist because the optimal utility is 0). Thus, the only possibility at equilibrium is $s_i(v) \leq \gamma^{(k)}$ almost surely.

- (ii) Proof of Part (b): Consider a value $v > \gamma^{(k)}$. Consider a value $v > \gamma^{(k)}$. We will show that the bid $b = \frac{v + \gamma^{(k)}}{2}$ gives the agent a strictly positive utility. Observe that the bid $b > \gamma^{(k)}$ and has allocation $x_i(b) > 0$ because the bid is strictly greater than $\gamma_{-i}^{(k)}$. To be precise, $\gamma_{-i}^{(k)} \leq \gamma^{(k)}$ for agent i . Since the bid $b < v$, the utility for this bid $\mathcal{U}_i(v, b) > 0$. The optimal bid can only do better. Thus, the optimal utility $\mathcal{U}_i(v) > 0$. Moreover, by definition of γ_i , the equilibrium bid satisfies $\gamma^{(k)} \leq s_i(v) < v$ almost surely.

The statement of the result holds. □

Furthermore, we reference a result from [Chawla and Hartline \[2013\]](#), which establishes that an agent obtains strictly higher utility with a larger value, and, almost surely, submits a higher bid. For completeness, we include a proof of this result in the appendix.

Lemma 25 ([Chawla and Hartline, 2013](#)). *Consider values v, w in the support of value distribution V_i such that $v > w > \gamma^{(k)}$. The following properties hold:*

- (a) *The optimal utility at these values is ordered as*

$$\mathcal{U}_i(v) > \mathcal{U}_i(w) > 0.$$

- (b) *The equilibrium bids are ordered as $s_i(v) \geq s_i(w)$ almost surely over the random strategy $s_i(\cdot)$.*

Proof. (i) Proof of Part (a): At value v , agent i could bid according to random strategy $s_i(w)$. Because the allocation is exactly what the agent would get if the value were w , such a strategy would give a strictly higher utility than $\mathcal{U}_i(w)$. The optimal strategy for value v can only do weakly better and thus $\mathcal{U}_i(v) > \mathcal{U}_i(w)$. The fact that both of these are strictly greater than 0 follows from [Lemma 24](#).

(ii) Proof of Part (b): The proof of this is exactly as that in the 1-unit setting.

The statement of the result holds. □

D.2.2 An Application of Neighborhood Deviation Arguments

This section establishes our main continuity and monotonicity result. We utilize a standard neighborhood deviation argument (see [Jin and Lu \[2023\]](#)) as a basis for the rest of our proof. They establish the continuity and monotonicity of equilibrium bid distributions in the single-unit setting. A key advantage of that setting is it only requires analyzing the highest equilibrium bid, which corresponds to the first-order statistic.

In the multi-unit setting however, a fundamental challenge arises: changes in an agent's equilibrium bid distribution need not necessarily impact the competing bid distribution of another agent. This is because an agent's allocation is determined solely by the k^{th} -highest bid among the remaining agents. Addressing this issue requires a different set of arguments. Before we prove the main result of this section, we provide the neighborhood deviation argument for completeness.

Lemma 26 (Neighborhood Deviation Argument, [Jin and Lu, 2023](#)). *Consider any agent i with interim allocation rule $x_i(\cdot)$. The following conditions are satisfied for any bid b :*

- (a) *Suppose there is a sequence of bids $(b_m)_{m=1}^{\infty}$ such that $\lim_{m \rightarrow \infty} b_m = b$. Moreover, there is a $\delta > 0$ for which $x_i(b_m) \geq x_i(b) + \delta$ for any m . Then b is not an equilibrium bid.*
- (b) *Suppose there is a sequence of bids $(b_m)_{m=1}^{\infty}$ and a $\delta > 0$ such that $b - b_m \geq \delta$ for every m . Moreover, the allocation probability $\lim_{m \rightarrow \infty} x_i(b_m) = x_i(b)$. Then b is not an equilibrium bid.*

Proof. We first prove part (a). Take a bid from the sequence $(b_m)_{m=1}^\infty$ that is arbitrarily close to b . This would give an agent strictly better utility than bidding b because $x_i(b_m) \geq x_i(b) + \delta$ for $\delta > 0$.

We next prove part (b). Take a bid from the sequence $(b_m)_{m=1}^\infty$ such that its allocation probability is arbitrarily close to $x_i(b)$. This would give an agent strictly better utility than bidding b because $b - b_m \geq \delta$ but their allocation probabilities are arbitrarily close. $\square$

We establish a key result below demonstrating that at most one agent can have a probability mass at any bid within the interval $(\gamma^{(k)}, \lambda^{(k)})$. The argument used to prove this result will serve as a foundation for several proofs later on.

Lemma 27. *Consider any bid b in $(\gamma^{(k)}, \lambda^{(k)})$. The bid distribution of at most 1 agent places strictly positive probability mass at b .*

Proof. Suppose there are 2 agents, i and j , such that their bid distributions $\mathcal{E}_i$ and $\mathcal{E}_j$ have probability mass at b . We will show that the interim allocation for one of these agents has a discontinuity at b . A sufficient condition for this is if the following properties hold:

- (i) With strictly positive probability, b is the bid of the k^{th} highest agent.
- (ii) With strictly positive probability, exactly $k - 1$ agents other than i and j bid strictly above b .

The rest of the proof will focus on showing the properties above. Define L as

$$L := \{l : \mathcal{E}_l(b) = 0\}.$$

The agents in L bid above b almost surely. It must be the case that $|L| \leq k - 1$ otherwise we would have $\gamma^{(k)} \geq b$ which is not possible. Next, define T as

$$T := \{l : \lambda_l = \lambda^{(k)}\}.$$

An application of [Lemma 22](#) tells us that $|T| \geq k + 1$. Consider a set T' such that

$$T' \subseteq T \setminus (\{i, j\} \cup L) \quad \text{and} \quad |T'| = k - 1 - |L|.$$

Such a set exists because $|T| \geq k + 1$ and $|L| \leq k - 1$. Consider the set $\mathcal{S} = L \cup T'$. We will show that agents in $\mathcal{S}$ bid above b with strictly positive probability. By construction, the size of $\mathcal{S}$ is $k - 1$. The agents in L bid above b almost surely. Any agent $l \in T'$ has $\lambda_l > b$ and thus bids above b with strictly positive probability. It follows that agents in $\mathcal{S}$ bid above b with strictly positive probability.

Now, any agent l that does not belong to $\mathcal{S}$ has $\mathcal{E}_l(b) > 0$. Thus, agents that do not belong to the set $\mathcal{S}$ bid at most b with strictly positive probability. Since agents i and j have probability mass at b , it is the bid of the k^{th} highest agent with strictly positive probability. Moreover, exactly $k - 1$ agents bid strictly above b with positive probability. The statement follows. $\square$

We establish one of the central results of this section: the distribution of the k^{th} highest equilibrium bid, $\mathcal{B}^{(k)}$, has a strictly increasing CDF in the closed interval $[\gamma^{(k)}, \lambda^{(k)}]$. The proof proceeds by induction on the agents, showing that the distribution cannot remain constant over any open subinterval of $[\gamma^{(k)}, \lambda^{(k)}]$.

Theorem 12. *The distribution of the k^{th} highest bid $\mathcal{B}^{(k)}$ has a strictly increasing CDF on the interval $[\gamma^{(k)}, \lambda^{(k)}]$.*

Proof. We will first show that the bid distribution of the k^{th} highest agent $\mathcal{B}^{(k)}$ is strictly increasing on the interval $[\gamma^{(k)}, \lambda^{(k)}]$. We will state an important lemma that will be subsequently used in our proof.

Lemma 28. *Suppose the bid distribution of the k^{th} highest agent $\mathcal{B}^{(k)}$ is flat in (α, β) . Then, no agent has a point mass in their bid distribution at any bid in (α, β) .*

Proof. Suppose agent i has a point mass at some bid b in the interval (α, β) . We will follow the approach in the proof of [Lemma 27](#). Our goal is to show that with strictly positive

probability there are exactly $k - 1$ agents who bid strictly above b and the rest bid strictly below b . This will imply that the bid of the k^{th} highest agent is b with strictly positive probability which is a contradiction to the assumption that $\mathcal{B}^{(k)}$ is flat in (α, β) .

Define L as

$$L := \{l : \mathcal{E}_l(b) = 0\}.$$

The agents in L bid above b almost surely. It must be the case that $|L| \leq k - 1$ otherwise we would have $\gamma^{(k)} \geq b$ which is a contradiction to $\gamma^{(k)} < \alpha$. Next, define T as

$$T := \{l : \lambda_l = \lambda^{(k)}\}.$$

An application of [Lemma 22](#) tells us that $|T| \geq k + 1$. Consider a set T' such that

$$T' \subseteq T \setminus (\{i\} \cup L) \quad \text{and} \quad |T'| = k - 1 - |L|.$$

Such a set exists because $|T| \geq k + 1$ and $|L| \leq k - 1$. We now consider the set $\mathcal{S} = L \cup T'$. We will show that agents in $\mathcal{S}$ bid above b with strictly positive probability. By construction, the size of $\mathcal{S}$ is $k - 1$. The agents in L bid above b almost surely. Consider any agent l in T' . This agent would bid strictly above b with positive probability because $\lambda_l > b$. It follows that agents in $\mathcal{S}$ bid above b with strictly positive probability.

Now, any agent l that does not belong to $\mathcal{S}$ has $\mathcal{E}_l(b) > 0$. Thus, agents that do not belong to the set $\mathcal{S}$ bid at most b with strictly positive probability. Since agent i has a probability mass at b , it is the bid of the k^{th} highest agent with strictly positive probability.

This is a contradiction to the assumption that $\mathcal{B}^{(k)}$ is flat in (α, β) . The statement of the result follows. $\square$

Suppose $\mathcal{B}^{(k)}$ has no probability density or mass on an open interval (α, β) . It is without loss for the interval to be maximal in the sense that for any $\delta > 0$, $\mathcal{B}^{(k)}$ has probability density or mass in $(\alpha - \delta, \alpha]$ and $[\beta, \beta + \delta)$. Thus, we have that $\mathcal{B}^{(k)}(\beta) \geq \mathcal{B}^{(k)}(\alpha) > 0$. Consider the following cases:

- (i) *Case 1:* Suppose $\mathcal{B}^{(k)}(\beta) > \mathcal{B}^{(k)}(\alpha) > 0$. It must be the case that $\mathcal{B}^{(k)}$ has a point mass at β . And thus, there is some agent i_0 that has probability mass at β in their equilibrium bid distribution. By Lemma 27, no other agent has a point mass at β in their bid distribution. The competing bid distribution $\mathcal{B}_{-i_0}^{(k)}$ is thus continuous at β .

If it is the case that agent i_0 's competing bid distribution $\mathcal{B}_{-i_0}^{(k)}$ has no probability density or probability mass in the open interval (α, β) , then agent i_0 's interim allocation is constant in $(\alpha, \beta]$. By Lemma 26, it is suboptimal for i_0 to have probability mass at β which is a contradiction to $\beta \in \text{supp}(\mathcal{E}_{i_0})$.

Thus, it must be that $\mathcal{B}_{-i_0}^{(k)}$ has probability density or probability mass at some point in (α, β) . But, because the distribution of the k^{th} highest bid was flat in (α, β) it must be the case that

$$\gamma_{i_0} \geq \sup \left\{ (\alpha, \beta) \cap \text{supp} \left(\mathcal{B}_{-i_0}^{(k)} \right) \right\}. \quad (\text{D.47})$$

This is because, if it were the case that $\gamma_{i_0} < \sup \left\{ (\alpha, \beta) \cap \text{supp} \left(\mathcal{B}_{-i_0}^{(k)} \right) \right\}$, then the bid of the k^{th} highest agent apart from i_0 becomes the bid of the k^{th} highest agent overall with strictly positive probability. Moreover, such bids would lie in (α, β) which is a contradiction to the assumption that the distribution of the k^{th} highest bid overall is flat in (α, β) .

Recall, we are analyzing a scenario in which $\mathcal{B}_{-i_0}^{(k)}$ has probability density or probability mass at some point in (α, β) . There must be an agent i_1 (clearly not the same as i_0 because we are analyzing the competing bid distribution) and a corresponding bid $\beta_1 \in (\alpha, \beta)$ such that i_1 has probability density on β_1 and $\gamma_{i_0} \geq \beta_1$ (an application of Lemma 28 tells us that this agent will not have a probability mass at β_1). Moreover, since no agent has a point mass at β_1 , as shown in Lemma 28, agent i_1 's competing bid distribution is continuous at β_1 .

We will inductively apply the above argument to show there are k agents whose $\gamma_i > \alpha$. This is a contradiction to $(\alpha, \beta) \subset [\gamma^{(k)}, \lambda^{(k)}]$.

Inductive hypotheses: Suppose we have a set of agents $\mathcal{I}_l = \{i_0, \dots, i_{l-1}\}$, where $|\mathcal{I}_l| <$

k . In addition, suppose this set satisfies the conditions stated below:

- (a) There is a corresponding set of bids $\mathbb{B}_l := \{\beta_j\}_{j=0}^{l-1}$ where $\beta_0 = \beta$ and $\beta_j \in (\alpha, \beta)$ for any $j \in [l-1]$. Moreover, the bids are monotonically decreasing, that is,

$$\beta_j > \beta_{j+1} \quad \text{for all } j \in \{0, \dots, l-1\}.$$

- (b) For every agent in $\mathcal{I}_l$, the competing bid distribution $\mathcal{B}_{-i_j}^{(k)}$ has probability density at some bid in (α, β_j) .

- (c) For every agent i_j in $\mathcal{I}_l$, the infimum of the support of the equilibrium bid distribution satisfies

$$\gamma_{i_j} \geq \sup \left\{ (\alpha, \beta_j) \cap \text{supp} \left(\mathcal{B}_{-i_j}^{(k)} \right) \right\}.$$

- (d) For any agent i_j , where $j \in \{0, \dots, l-2\}$,

$$\gamma_{i_j} \geq \beta_{j+1}.$$

- (e) There is an agent $i_l \notin \mathcal{I}_l$ and a bid β_l in its support such that

$$\alpha < \beta_l < \beta_{l-1} \text{ and } \gamma_{i_{l-1}} \geq \beta_l.$$

Base case: Recall, we have shown that the conditions above are satisfied for $l = 1$. The set in consideration is $\mathcal{I}_l = \{i_0\}$. Part (a) is satisfied vacuously with the set of bids $\{\beta\}$. Part (b) is satisfied because we are analyzing the scenario where $\mathcal{B}_{-i_0}^{(k)}$ has probability density or probability mass in (α, β) . Part (c) is satisfied by (D.47). Part (d) holds vacuously. Part (e) is satisfied by agent i_1 : there is a bid $\beta_1 \in \text{supp}(\mathcal{E}_{i_1})$ such that $\beta_1 \in (\alpha, \beta)$ and $\gamma_{i_0} \geq \beta_1$.

Inductive argument: Suppose the conditions hold for some l where $1 < l < k$. We will analyze the competing bid distribution of agent i_l . Consider the case $\mathcal{B}_{-i_l}^{(k)}$ is flat in

(α, β_l) . By Lemma 28, no agent has a point mass at β_l and thus $\mathcal{B}_{-i_l}^{(k)}$ is continuous at β_l . An immediate consequence of this is i_l has a constant allocation rule in the interval $(\alpha, \beta_l]$. But, by Lemma 26, this is a contradiction to the fact that $\beta_l \in \text{supp}(\mathcal{E}_{i_l})$.

Thus, it must be the case that $\mathcal{B}_{-i_l}^{(k)}$ has density at some point in (α, β_l) (at the expense of being repetitive, by Lemma 28, $\mathcal{B}_{-i_l}^{(k)}$ cannot have a point mass in this interval). Let us define the set

$$\mathcal{I}_{l+1} := \mathcal{I}_l \uplus \{i_l\}.$$

By Part (e), $i_l \notin \mathcal{I}_l$, and thus you are adding a unique element to $\mathcal{I}_l$. Let us also consider the corresponding set of bids

$$\mathbb{B}_{l+1} := \{\beta_j\}_{j=0}^{l-1} \cup \{\beta_l\}.$$

By Part (e), $\beta_l < \beta_{l-1}$ and thus, applying Part (a),

$$\beta_j > \beta_{j+1} \text{ for all } j \in \{0, \dots, l\}.$$

Thus $\mathcal{I}_{l+1}$ and $\mathbb{B}_{l+1}$ satisfies Part (a) for $l+1$ agents. Part (b) is satisfied for $(l+1)$ agents as well because we are analyzing the case $\mathcal{B}_{-i_l}^{(k)}$ has density at some point in (α, β_l) .

We will now show that Part (c), (d) and (e) are also satisfied for $(l+1)$ agents. The competing bid distribution $\mathcal{B}_{-i_l}^{(k)}$ has density at some point in (α, β_l) . However, the distribution of the bid of the k^{th} highest agent is flat in (α, β) . Thus, it must be the case that

$$\gamma_{i_l} \geq \sup \left\{ (\alpha, \beta_l) \cap \text{supp} \left(\mathcal{B}_{-i_l}^{(k)} \right) \right\}. \quad (\text{D.48})$$

This is because, if it were the case that $\gamma_{i_l} < \sup \left\{ (\alpha, \beta_l) \cap \text{supp} \left(\mathcal{B}_{-i_l}^{(k)} \right) \right\}$, then the bid of the k^{th} highest agent apart from i_l becomes the bid of the k^{th} highest agent overall with strictly positive probability. Moreover, such bids would lie in (α, β) which is a

contradiction to the assumption that the distribution of the k^{th} highest bid overall is flat in (α, β) . Since, by the inductive hypotheses, this condition also holds for agents $j \in \{0, \dots, l-1\}$, Part (c) is satisfied for $(l+1)$ agents. Next, Part (d) is satisfied for $(l+1)$ agents because in Part (e) of the induction hypotheses, we had that $\gamma_{i_{l-1}} \geq \beta_l$.

Now, $\mathcal{B}_{-i_l}^{(k)}$ has density at some point in (α, β_l) . Thus, there must exist an agent i_{l+1} and a corresponding bid $\beta_{l+1} \in (\alpha, \beta_l)$ such that $\beta_{l+1} \in \text{supp}(\mathcal{E}_{i_{l+1}})$. In addition to this, an implication of (D.48) is that $\gamma_{i_l} \geq \beta_{l+1}$. Finally, the induction hypotheses tells us that for any agent in $\mathcal{I}_l$, the infimum of the support of their bid distribution

$$\gamma_{i_j} \geq \beta_{j+1} \geq \beta_l.$$

Thus it cannot be that agent i_{l+1} , who has probability density in (α, β_l) , lies in the set $\mathcal{I}_l$. This satisfies Part (e).

To summarize the conclusions, at each index $l \in [k-1]$, either we have a contradiction in the case $\mathcal{B}_{-i_l}^{(k)}$ is flat in (α, β_l) or the conditions of the inductive hypotheses are satisfied for index $(l+1)$. If the latter is satisfied at each index, we have a set of k agents $\mathcal{I}_k$ such that

$$\gamma_{i_j} \geq \beta_{j+1} > \alpha \quad \text{for all } j \in \{0, \dots, k-1\}.$$

But, this is a contradiction to the infimum of bid distribution of the k^{th} highest agent $\gamma^{(k)}$ being strictly smaller than α .

- (ii) *Case 2:* Suppose $\mathcal{B}^{(k)}(\beta) = \mathcal{B}^{(k)}(\alpha) > 0$. First, we will show that no agent can have a point mass at β . To see this, consider the argument in the proof of Lemma 28. Suppose there exists an agent, labeled i , who has a point mass at β . Define L as

$$L := \{l : \mathcal{E}_l(\beta) = 0\}.$$

The agents in L bid above β almost surely. It must be the case that $|L| \leq k-1$ otherwise we would have $\gamma^{(k)} \geq \beta$ which is a contradiction to $\gamma^{(k)} < \alpha$. Next, define T

as

$$T := \{l : \lambda_l = \lambda^{(k)}\}.$$

An application of [Lemma 22](#) tells us that $|T| \geq k + 1$. Consider a set T' such that

$$T' \subseteq T \setminus (\{i\} \cup L) \quad \text{and} \quad |T'| = k - 1 - |L|.$$

Such a set exists because $|T| \geq k + 1$ and $|L| \leq k - 1$. We now consider the set $\mathcal{S} = L \uplus T'$. We will show that agents in $\mathcal{S}$ bid above β with strictly positive probability. By construction, the size of $\mathcal{S}$ is $k - 1$. The agents in L bid above β almost surely. Any agent $l \in T'$ has $\lambda_l > \beta$. Thus, he bids strictly above β with positive probability. It follows that agents in $\mathcal{S}$ bid above β with strictly positive probability.

Now, any agent l that does not belong to $\mathcal{S}$ has $\mathcal{E}_l(\beta) > 0$. Thus, agents that do not belong to the set $\mathcal{S}$ bid strictly smaller than β with positive probability. It follows that β is the bid of the k^{th} highest agent with positive probability because agent i bids β with positive probability. This is a contradiction to the assumption that $\mathcal{B}^{(k)}(\alpha) = \mathcal{B}^{(k)}(\beta)$. Thus, no agent can have a point mass at β .

We will show that the induction argument presented in *Case 1* extends to this case as well.

Recall, the interval (α, β) is maximal in the following sense: for any $\delta > 0$, $\mathcal{B}^{(k)}$ has probability density or mass in $(\alpha - \delta, \alpha]$ and $[\beta, \beta + \delta)$. Thus, there must exist an agent i_0 who has β in the support of his equilibrium bid distribution $\mathcal{E}_{i_0}$.

If it is the case that agent i_0 's competing bid distribution $\mathcal{B}_{-i_0}^{(k)}$ has no probability density or probability mass in the open interval (α, β) , then agent i_0 's interim allocation is constant in $(\alpha, \beta]$. This is because no agent has a point mass at β and thus $\mathcal{B}_{-i_0}^{(k)}$ is continuous at that point. By [Lemma 26](#), it is suboptimal for i_0 to have probability mass at β which is a contradiction to $\beta \in \text{supp}(\mathcal{E}_{i_0})$.

Thus, it must be that $\mathcal{B}_{-i_0}^{(k)}$ has probability density or probability mass at some point in (α, β) . However, because the distribution of the k^{th} highest bid was flat in (α, β) it

must be the case that

$$\gamma_{i_0} \geq \sup \left\{ (\alpha, \beta) \cap \text{supp} \left(\mathcal{B}_{-i_0}^{(k)} \right) \right\},$$

i.e. the inequality in D.47, holds.

Moreover, there must be an agent i_1 (clearly not the same as i_0 because we are analyzing the competing bid distribution) and a corresponding bid $\beta_1 \in (\alpha, \beta)$ such that i_1 has probability density on β_1 and $\gamma_{i_0} \geq \beta_1$ (an application of Lemma 28 tells us that this agent will not have a probability mass at β_1). Moreover, since no agent has a point mass at β_1 , as shown in Lemma 28, agent i_1 's competing bid distribution is continuous at β_1 .

The arguments above show that the base case for the induction argument is satisfied. The conclusion of the argument is the same and we thus have a contradiction.

Both of these cases imply that there does not exist an open interval in $[\gamma^{(k)}, \lambda^{(k)}]$ such that $\mathcal{B}^{(k)}$ has no probability density or mass in it. Thus, the bid distribution $\mathcal{B}^{(k)}$ is strictly increasing on $[\gamma^{(k)}, \lambda^{(k)}]$. $\square$

Theorem 13. *Consider any agent $i \in \mathcal{N}$. The competing bid distribution $\mathcal{B}_{-i}^{(k)}$ has a strictly increasing CDF on the interval $[\gamma^{(k)}, \lambda^{(k)}]$.*

Proof. Consider any agent $i \in \mathcal{N}$. Suppose the competing bid distribution $\mathcal{B}_{-i}^{(k)}$ is flat in an open interval $(\alpha, \beta) \subset [\gamma^{(k)}, \lambda^{(k)}]$. It is without loss for this interval to be maximal: for any $\delta > 0$, distribution $\mathcal{B}_{-i}^{(k)}$ has probability density or mass in $(\alpha - \delta, \alpha]$ and $[\beta, \beta + \delta)$. By Theorem 12, however, the bid of the k^{th} highest agent is strictly increasing in (α, β) . This implies there is a bid b in the interval (α, β) such that b is in the support of agent i 's equilibrium bid distribution.

Since $b \in (\alpha, \beta)$, the competing bid distribution is flat in $(\alpha, b]$. This means that agent i has a constant interim allocation $x_i(\cdot)$ in $(\alpha, b]$. By Lemma 26, this is a contradiction to $b \in \text{supp}(\mathcal{E}_i)$. Thus, the competing bid distribution $\mathcal{B}_{-i}^{(k)}$ is strictly increasing in $[\gamma^{(k)}, \lambda^{(k)}]$. $\square$

The previous results established that the distribution of the k^{th} highest bid $\mathcal{B}^{(k)}$ and the competing bid distribution $\mathcal{B}_{-i}^{(k)}$ are strictly increasing in the interval $[\gamma^{(k)}, \lambda^{(k)}]$. In addition to this, we determine these bid distribution are continuous on $(\gamma^{(k)}, \lambda^{(k)})$.

Theorem 14. *The bid distribution $\mathcal{B}^{(k)}$ of the k^{th} highest agent is continuous on the interval $(\gamma^{(k)}, \lambda^{(k)})$.*

Proof. Suppose $\mathcal{B}^{(k)}$ is discontinuous at some bid b^* in $(\gamma^{(k)}, \lambda^{(k)})$. There must be some agent i who has a point mass at b^* in the support of his equilibrium bid distribution. By Lemma 27, only agent i has a point mass at this bid. But since, by Theorem 13, his competing bid distribution $\mathcal{B}_{-i}^{(k)}$ is strictly increasing at b^* , there is a different agent i' such that b^* is in the support of the equilibrium bid distribution $\mathcal{E}_{i'}$. An argument similar to the one used in the proof of Lemma 27 will imply that the interim allocation $x_{i'}(\cdot)$ is discontinuous at b^* . This means there exists a $\delta' > 0$ such that

$$x_{i'}(b') \geq x_{i'}(b^*) + \delta' \quad \text{for all } b' > b^*. \quad (\text{D.49})$$

We will show that this is a contradiction to the following statement: there are bids smaller than b^* and arbitrarily close to it that lie in the support of the equilibrium distribution of some agent.

- (i) Suppose for any $\delta > 0$, the bid distribution $\mathcal{E}_{i'}$ has positive probability in $(b^* - \delta, b^*]$. Consider a sequence of bids $(b_l)_{l=1}^{\infty}$ such that each element b_l lies in the interval $(b^* - \frac{1}{l}, b^*)$ and $b_l \in \text{supp}(\mathcal{E}_{i'})$. The existence of such a bid b_l , for every $l \in \mathbb{N}$, is guaranteed by the assumption that the distribution has positive probability in the interval $(b^* - \delta, b^*]$ for every $\delta > 0$. The sequence (b_l) converges to b^* . Let v_l be the corresponding value for which b_l is an equilibrium bid. By Lemma 25, the sequence of values $(v_l)_{l=1}^{\infty}$ can be chosen to be weakly increasing. The utility of agent i' along this sequence is

$$(v_l - b_l)x_{i'}(b_l).$$

We will establish that the limit infimum of the sequence of values must be strictly

greater than b^* , that is,

$$\liminf_{l \rightarrow \infty} v_l > b^*.$$

If this was not the case, then large enough values in the sequence obtain equilibrium utility that is arbitrarily close to 0. This is a contradiction to the fact that equilibrium utility is monotonic in values and strictly positive as established in [Lemma 25](#). Thus, it must be that $\liminf_{l \rightarrow \infty} v_l > b^*$.

Because of the discontinuity in the allocation rule at b^* , it follows by [Lemma 26](#) that for values large enough in the sequence above, there exists a bid b' strictly greater than b^* which gives the agent a strictly better utility. This is a contradiction to bids being arbitrarily close to b^* belonging to the support of $\mathcal{E}_{i'}$.

- (ii) Suppose there exists a $\delta > 0$ such that $\mathcal{E}_{i'}$ has 0 probability in $(b^* - \delta, b^*]$. But because $b^* \in \text{supp}(\mathcal{E}_{i'})$ it must be that for any $\delta > 0$, bid distribution $\mathcal{E}_{i'}$ has positive probability in $[b^*, b^* + \delta)$. We will show there exists an agent $i^\dagger$, not the same as i and i' , such that for all $\delta_{i^\dagger} > 0$, he has positive probability in $(b^* - \delta_{i^\dagger}, b^*]$. Suppose that apart from i and i' , no agent satisfies the above condition. By assumption, there exists a $\delta > 0$ such that $\mathcal{E}_{i'}$ has 0 probability in $(b^* - \delta, b^*]$. Thus, the competing bid distribution of agent i $\mathcal{B}_{-i}^{(k)}$ is flat in $(b^* - \delta_i, b^*]$ for some $\delta_i > 0$. This is a contradiction to the result that $\mathcal{B}_{-i}^{(k)}$ is strictly increasing in the interval $[\gamma^{(k)}, \lambda^{(k)}]$ (which was stated as [Theorem 12](#)).

Thus, there exists an agent $i^\dagger$, not the same as i and i' , such that for all $\delta_{i^\dagger} > 0$, he has positive probability in $(b^* - \delta_{i^\dagger}, b^*]$. A consequence of this is that $b^* \in \text{supp}(\mathcal{E}_{i^\dagger})$. Moreover, this agent will have a discontinuous interim allocation rule at b^* . Applying the argument in Part (i), we get a contradiction to bids arbitrarily close to b^* belonging to the support of $\mathcal{E}_{i^\dagger}$.

Since we get a contradiction in both cases, it cannot be the case that $\mathcal{B}^{(k)}$ is discontinuous in $(\gamma^{(k)}, \lambda^{(k)})$. $\square$

Theorem 15. *Consider any agent $i \in \mathcal{N}$. The competing bid distribution $\mathcal{B}_{-i}^{(k)}$ is continuous on the interval $(\gamma^{(k)}, \lambda^{(k)})$. Moreover, the agents bid distribution $\mathcal{E}_i$ is also continuous in the*

interval $(\gamma^{(k)}, \lambda^{(k)})$.

Proof. Suppose $\mathcal{B}_{-i}^{(k)}$ is discontinuous at some bid b^* in the interval $(\gamma^{(k)}, \lambda^{(k)})$. There must be some agent j (not the same as i) who has a point mass at b^* in the support of his equilibrium bid distribution. The statement then follows from the argument given in the proof of [Theorem 14](#).

We will now analyze the bid distribution $\mathcal{E}_i$. Suppose the bid distribution is discontinuous at some bid b^* in the interval $(\gamma^{(k)}, \lambda^{(k)})$. Thus, agent i has a point mass at b^* . The continuity of $\mathcal{E}_i$ follows from the argument given in the proof of [Theorem 14](#). $\square$

The previous results established continuity of the bid distributions in the open interval $(\gamma^{(k)}, \lambda^{(k)})$. We extend this result to show that the distribution of the k^{th} highest bid and critical bid distributions $\mathcal{B}_{-i}^{(k)}$ for every agent i is continuous in the interval $(\gamma^{(k)}, \lambda^{(k)})$.

Lemma 29. *Consider any profile of equilibrium bid distributions $\{\mathcal{E}_i\}_{i \in \mathcal{N}}$. At most $k - 1$ agents have a point mass at $\lambda^{(k)}$ in their equilibrium bid distribution.*

Proof. We will first show at most k agents can have a point mass at $\lambda^{(1)}$ in their equilibrium bid distributions. Suppose at least $(k + 1)$ agents have a point mass at $\lambda^{(1)}$. We will show there must be an agent who has a discontinuity in their allocation at $\lambda^{(1)}$. Formally, there exists an agent i and a corresponding $\delta_i > 0$ satisfying

$$x_i(b) \geq x_i(\lambda^{(1)}) + \delta_i \quad \text{for all } b > \lambda^{(1)}.$$

The allocation probability of agent i for bid b is

$$x_i(b) = \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b] + \mathbb{P}[\mathcal{B}_{-i}^{(k)} = b] \mathbb{P}[\text{agent } i \text{ wins in a tie-break} \mid \mathcal{B}_{-i}^{(k)} = b].$$

Because at least $(k + 1)$ agents have a point mass at $\lambda^{(1)}$, and there are only k items available, there must exist an agent i for which

$$\mathbb{P}[\text{agent } i \text{ wins in a tie-break} \mid \mathcal{B}_{-i}^{(k)} = \lambda^{(1)}] < 1.$$

The allocation rule is discontinuous at $\lambda^{(1)}$ for this agent. A neighborhood deviation argument (as stated in Lemma 26) implies that $\lambda^{(1)}$ cannot be a best response bid of agent i . This is a contradiction to $\lambda^{(1)}$ being in the support of their equilibrium bid distribution. Thus, at most k agents can have a point mass at $\lambda^{(1)}$ in their equilibrium bid distribution.

Suppose exactly k agents have a point mass at $\lambda^{(1)}$. By Lemma 22, there exists an agent $i^\dagger$ such that the equilibrium bid distribution $\mathcal{E}_{i^\dagger}$ does not have a point mass at $\lambda^{(1)}$ but where $\lambda^{(1)}$ belongs to its support. It must be that

$$\mathbb{P}[\text{agent } i^\dagger \text{ wins in a tie-break} \mid \mathcal{B}_{-i^\dagger}^{(k)} = \lambda^{(1)}] < 1. \quad (\text{D.50})$$

If this is not the case, then there must exist an agent i^* who has a point mass at $\lambda^{(1)}$ in their equilibrium bid distribution and

$$\mathbb{P}[\text{agent } i^* \text{ wins in a tie-break} \mid \mathcal{B}_{-i^*}^{(k)} = \lambda^{(1)}] < 1.$$

However, this implies i^* has a discontinuity in their allocation rule at $\lambda^{(1)}$. Applying the neighborhood deviation argument, as was applied in the paragraph above, would lead to a contradiction. (D.50) implies the allocation rule of $i^\dagger$ is discontinuous at $\lambda^{(1)}$. This is a contradiction to the fact that $\lambda^{(1)}$ is in the support of $\mathcal{E}_{i^\dagger}$. Thus, at most $k - 1$ agents have a point mass at $\lambda^{(1)}$ in their equilibrium bid distribution. $\square$

A direct implication of the preceding result is the continuity of competing bid distributions on the interval $[\gamma^{(k)}, \lambda^{(k)}]$.

Theorem 16. *Consider any agent $i \in \mathcal{N}$. The competing bid distribution $\mathcal{B}_{-i}^{(k)}$ has a continuous cumulative distribution function on the interval $[\gamma^{(k)}, \lambda^{(k)}]$.*