Turbulent Heat Transfer to a Rotating Disk: a Review and Extension of Dorfman

M. C. Johnson[1]

Nomenclature

T = temperature
T_w = temperature of disk wall
T_∞ = temperature of rotating fluid core
R = radius
Z = axial distance from rotating disk
W = angular velocity of disk
q = heat flux
V_R = velocity of fluid in radial direction
$V_{R\mathrm{AVE}}$ = average radial velocity at a particular radius
V_∞ = tangential velocity of fluid in core
V_Z = velocity of fluid in axial direction
δ_T = thermal boundary layer size
δ = momentum boundary layer size
ρ = fluid density
c_p = fluid heat capacity
$Pr = \mu c_\rho / K$ (Prandtl number)
μ = fluid viscosity
h = film coefficient
$\mathrm{Re} = (WR - V_\infty)R\rho/\mu$ (Reynolds number)
St = Nu/Pr Re (Stanton number)
ν = kinematic viscosity

Introduction

Heat transfer from a rotating disk to the rotating fluid adjacent to it is defined by the boundary layer energy equation

$$\rho c_p \left(V_R \frac{\partial T}{\partial R} + V_Z \frac{\partial T}{\partial z} \right) = \frac{\partial q}{\partial z} \tag{1}$$

Normally, the heat flux at the disk wall is found using Reynolds analogy. Reynolds analogy applies when the form of the tangential momentum equation and the energy equation are shown to be exactly similar. As shown by Dorfman [1], the Reynolds analogy also requires the tangential momentum and energy boundary conditions to be exactly similar. For a rotating disk, these conditions can be generalized as follows [2].

$$\frac{\partial T_\infty/\partial R}{T_w - T_\infty} = \frac{\partial/\partial R(RV_\infty)}{R^2W - RV_\infty} \tag{2}$$
$$\frac{\partial/\partial R(T_w - T_\infty)}{T_w - T_\infty} = \frac{\partial/\partial R(R^2W - RV_\infty)}{R^2W - RV_\infty}$$

For the case of a disk rotating in a quiescent atmosphere, these conditions become

$$T_w - T_\infty = C_0R^2$$
$$T_\infty = C_1 \tag{3}$$

In general, these boundary conditions are not met, and it is necessary to look elsewhere for a solution of the boundary layer energy equation.

Dorfman's Method

Dorfman [1] considered the case of a disk rotating in a quiescent atmosphere ($V_\infty = 0$). He integrated the boundary layer energy equation (1) across the thermal boundary layer and introduced the continuity equation, yielding

$$d/dR \left[R \int_0^{\delta_T} V_R(T - T_\infty)dz \right] = -\left.\frac{Rq}{\rho c_p}\right|_{z=0} \tag{4}$$

or, in the dimensionless form

$$\frac{1}{WR^2(T_w - T_\infty)}\, d/dR \left[R \int_0^{\delta_T} V_R(T - T_\infty)dz \right] = \mathrm{St} \tag{4a}$$

The primary difficulty in solving this equation is the assumption of a model for the wall heat flux, which Dorfman assumes in the following form

$$q|_{z=0} = \text{constant} \cdot \left[\int_0^{\delta_T} V_R(T - T_\infty)dz \right]^{-m} \tag{5}$$

where m is a constant to be determined experimentally.

The assumption of (5) reduces equation (4) to a separable nonlinear differential equation.

The actual form of Dorman's assumption is

$$\mathrm{St}^{-1} = A(\mathrm{Pr})(\mathrm{Re}^{**})^m \tag{6}$$

$A(\mathrm{Pr})$ is a function of the Prandtl number and Re^{**} is a thermal Reynolds number using a weighted thermal boundary layer size as a characteristic dimension. It is defined in the next section.

Equation (6) can be rearranged in the form

$$q = \frac{\rho c_p WR(T_w - T_\infty)}{A(\mathrm{Pr})} \left[\frac{\rho}{\mu(T_w - T_\infty)} \int_0^{\delta_T} V_R(T - T_\infty)dz \right]^{-m} \tag{7}$$

Substituting this expression into equation (4), separating variables, assuming constant fluid properties and integrating, an expression for the Stanton number can be found.

$$\mathrm{St} = \frac{\mathrm{Re}^{-m/m+1}(T_w - T_\infty)^m R^{m(m+3)/m+1}}{[A(\mathrm{Pr})]^{1/m+1}} \times \left[\int_0^R (T_w - T_\infty)^{m+1} R^{m+2} dR \right]^{-m/m+1} \tag{8}$$

This equation represents a general expression for the Stanton number. Now m and $A(\mathrm{Pr})$ must be determined empirically. Although any data correlation could be used, it will be convenient to use one such that the disk temperature distribution is rigidly defined. Such conditions exist for the Reynolds analogy.

Dorfman gives an expression for the Stanton number of a disk rotating in a quiescent atmosphere for Reynolds analogy conditions (turbulent flow) [1].

$$\mathrm{St} = 0.0267\ \mathrm{Pr}^{-0.4}\mathrm{Re}^{-0.2} \tag{9}$$

As shown in equation (3), the Reynolds analogy temperature distribution on a disk rotating in a quiescent atmosphere is parabolic ($T_w = C_0R^2 + T_\infty$). Substituting this expression for T_w into equation (8), and rearranging in the form of equation (9), the required empirical constants are found to be

$$m = 0.25$$
$$A(\mathrm{Pr}) = 135.7\ \mathrm{Pr}^{0.5} \tag{10}$$

Therefore, for turbulent flow conditions, the expression for the Stanton number for a rotating disk in a quiescent atmosphere is

$$\mathrm{St} = \frac{\mathrm{Pr}^{-0.4}\mathrm{Re}^{-0.2}T_w - T_\infty)^{0.25}R^{0.65}}{53.14 \left[\int_0^R (T_w - T_\infty)^{1.25} R^{2.25} dR \right]^{0.2}} \tag{11}$$

An application of this equation for temperature distributions of the form: $T_w - T_\infty = BR^m$ is discussed in reference [3].

Improved Method

Arbitrary Fluid Core Rotation and Density Variations. Dorfman considered only the case of a disk rotating in a quiescent atmosphere. He defined the thermal Reynolds number as

[1] Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.

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$$\mathrm{Re}^{**} = \frac{WR\delta_T^{**}}{\nu}$$

and the integral thermal boundary layer thickness as

$$\delta_T^{**} = \int_0^{\delta_T} \frac{V_R(T - T_\infty)dz}{WR(T_w - T_\infty)}$$

A simple extension of Dorfman's method can be effected by allowing for arbitrary fluid core rotation. This can be done by replacing the wheel speed (WR) with the relative speed ($WR - V_\infty$). The thermal Reynolds number then becomes

$$\mathrm{Re}^{**} = \frac{(WR - V_\infty)\delta_T^{**}}{\nu}$$

and the integral thermal boundary layer thickness becomes

$$\delta_T^{**} = \int_0^{\delta_T} \frac{V_R(T - T_\infty)dz}{(WR - V_\infty)(T_w - T_\infty)}$$

Using these forms in equation (6) and allowing for density variations, equation (11) for the Stanton number becomes

$$\mathrm{St} = \frac{\mathrm{Pr}^{-0.4}(T_w - T_\infty)^{0.25}R^{0.25}\mu^{0.2}}{(53.14)\left[\int_0^R \rho R^{1.25}(WR - V_\infty)(T_w - T_\infty)^{1.25}dR\right]^{0.2}} \quad (12)$$

Beginning of Boundary Layer Growth. Dorfman assumes that boundary layer growth begins at $R = 0$. This is not always the case. A common example can be found in the compressor disk cavities of a gas turbine engine. Mid-compressor air is bled into these cavities for cooling purposes. The air enters at the outer diameter of the disk cavities and exits at the bores in the center of the disks. On the disks, a boundary layer develops which begins at the outer diameter, flows radially inward and exits at the disk bores.

This modifies equation (12) as follows

$$\mathrm{St} = \frac{\mathrm{Pr}^{-0.4}(T_w - T_\infty)^{0.25}R^{0.25}\mu^{0.2}}{(53.14)\left[\int_{R_0}^R \rho R^{1.25}(WR - V_\infty)(T_w - T_\infty)^{1.25}dR\right]^{0.2}} \quad (13)$$

R_0 is the radius where boundary layer growth begins.

Variations in Fluid Core Temperature. If the fluid core temperature (T_∞) cannot be considered a constant, equation (4) needs to be modified

$$d/dR\left[\rho R \int_0^{\delta_T} V_R(T - T_\infty)dz\right] + R\rho \int_0^{\delta_T} V_R \frac{dT_\infty}{dR} dz = -\left.\frac{Rq}{c_p}\right|_{z=0} \quad (14)$$

or, as in equation (4a), in terms of Stanton number

$$\frac{1}{R(T_w - T_\infty)(WR - V_\infty)\rho}\left[d/dR\left(\rho R \int_0^{\delta_T} V_R(T - T_\infty)dz\right) + R\rho \int_0^{\delta_T} V_R \frac{dT_\infty}{dR} dz\right] = \mathrm{St} \quad (14a)$$

Substitution of assumption (7) into equation (14) will lead to a differential equation which is no longer separable. To solve the resulting equation it is necessary to make several new assumptions.

1 Assume a shape for the radial velocity profile in the boundary layer:

$$V_R = 2.45 V_{R_{\mathrm{AVE}}}(Z/\delta)^{1/7}[1 - (Z/\delta)] \quad (15)$$

2 Assume a shape for the temperature profile in the boundary layer:

$$T = T_w + (T_\infty - T_w)(Z/\delta)^{1/7} \quad (16)$$

3 Assume that the thermal boundary layer size is greater than or equal to the momentum boundary layer size:

$$\delta_T \geq \delta \quad (17)$$

Using these assumptions along with equations (7) and (14) yields

$$\eta^m\left[\frac{d\eta}{dR} + \frac{k\eta}{T_w - T_\infty}\frac{dT_\infty}{dR}\right] = \frac{\rho^{m+1}\nu^m(T_w - T_\infty)^{m+1}(RW - V_\infty)R^{m+1}}{A(\mathrm{Pr})} \quad (18)$$

where

$$\eta = R\rho(RW - V_\infty)(T_w - T_\infty)\delta_T^{**}$$

k is a constant arising from the integration of the product of the radial velocity and temperature boundary layer profiles.

This equation can be solved by the introduction of an integrating factor

$$\exp\left[(m+1)k \int \frac{dT_\infty/dR}{T_w - T_\infty} dR\right]$$

Using the integrating factor, equation (18) can be integrated. Following the technique used to derive equation (11) yields equation (19).

For temperature profiles whose shape is known to be different from that assumed in equation (16), equation (19) can be changed to reflect this by changing the constant k introduced in equation (18).

The only experimental results available are for the isothermal disk rotating in a constant temperature environment [4]. For that case, equation (19) reduces to match the experimental results.

References

1 Dorfman, L. A., *Hydrodynamic Resistance and the Heat Loss of Rotating Solids*, pp. 87–98, Oliver and Boyd, 1963.

2 Lord, W., personal communication.

3 Harnett, J. P., Shing-Hwa Tsai, H. N. Jantscher, "Heat Transfer to a Nonisothermal Disk with a Turbulent Boundary Layer", ASME JOURNAL OF HEAT TRANSFER, Vol. 87, 1965, pp. 362–368.

4 Kreith, F. *Principles of Heat Transfer*, Intext Educational Publishers, 1973, pp. 404, 405.

$$\mathrm{St} = \frac{\mathrm{Pr}^{-0.4}R^{0.25}(T_w - T_\infty)^{0.25}\mu^{0.2}\exp\left(1.22\int \frac{dT_\infty/dR}{T_w - T_\infty}dR\right)}{(53.14)\left[\int_{R_0}^R \rho R^{1.25}(T_w - T_\infty)^{1.25}(WR - V_\infty)\exp\left(6.1\int \frac{dT_\infty/dR}{T_w - T_\infty}dR\right)dR\right]^{0.2}} \quad (19)$$