Investigation of Crouzeix's Conjecture via Nonsmooth Optimization

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Joint work with
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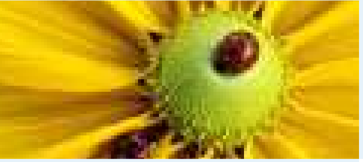
Crouzeix's Conjecture

The Field of Values

For $A \in \mathbb{C}^{n \times n}$, the field of values (or numerical range) of A is

$$W(A) = \{v^* A v : v \in \mathbb{C}^n, \|v\|_2 = 1\} \subset \mathbb{C}.$$



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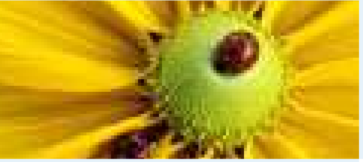

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Toeplitz-Haussdorf Theorem: $W(A)$ is convex for all $A \in \mathbb{C}^{n\times n}$.

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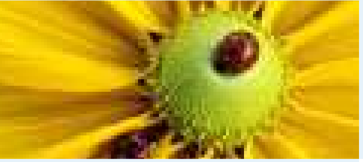

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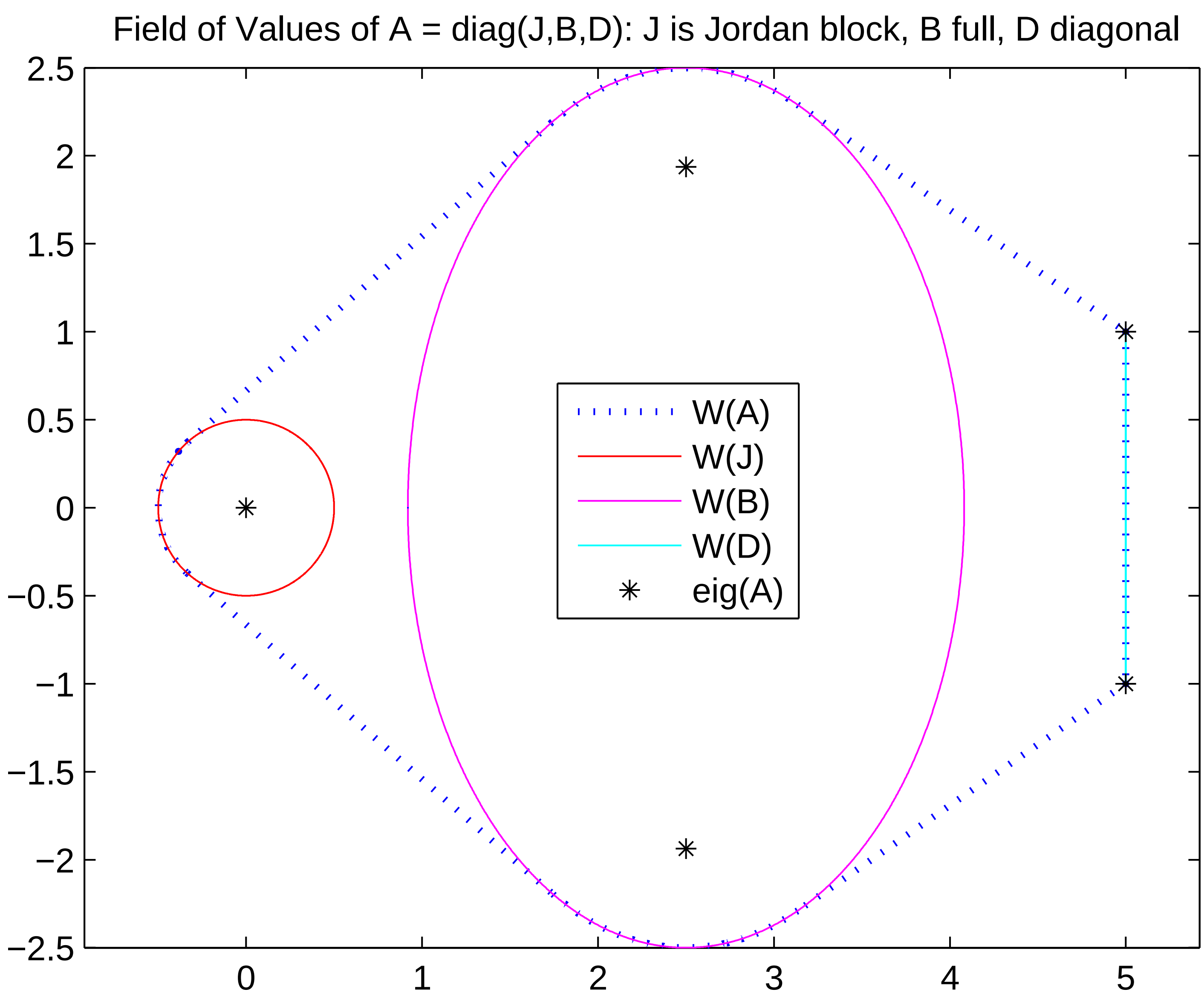
$$A = \operatorname{diag}(J, B, D) : \quad W(A) = \operatorname{conv}\left(W(J), W(B), W(D)\right)$$

Example, continued



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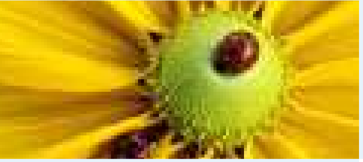
If $p = \chi(A)$, the characteristic polynomial (or minimal polynomial) of A, then $\|p(A)\|_2 = 0$ by Cayley-Hamilton, but $\|p\|_{W(A)} = 0$ only if $A = \lambda I$ for $\lambda \in \mathbb{C}$, so that $W(A) = \{\lambda\}$.



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If $p(\zeta) = \zeta$ and A is a 2×2 Jordan block with 0 on the diagonal, then $\|p(A)\|_2 = 1$ and $W(A)$ is a disk centered at 0 with radius 0.5, so the left and right-hand sides are equal.

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$$\|p(A)\|_2 \leq 11.08\, \|p\|_{W(A)}$$

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Palencia's theorem (2016)

$$\|p(A)\|_2 \leq \left(1+\sqrt{2}\right)\|p\|_{W(A)}$$

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Published in SIMAX, May 2017, with Crouzeix.

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The conjecture is known to hold for certain restricted classes of polynomials $p \in P^m$ or matrices $A \in \mathbb{C}^{n \times n}$.
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- $AA^* = A^*A$ (then the constant 2 can be improved to 1).

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where v_θ is a normalized eigenvector corresponding to the largest eigenvalue of the Hermitian matrix

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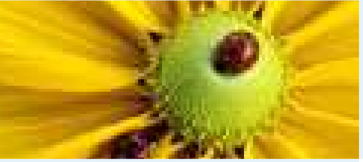

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Thus, we can compute as many extreme points as we like.

Continuing with the previous example...

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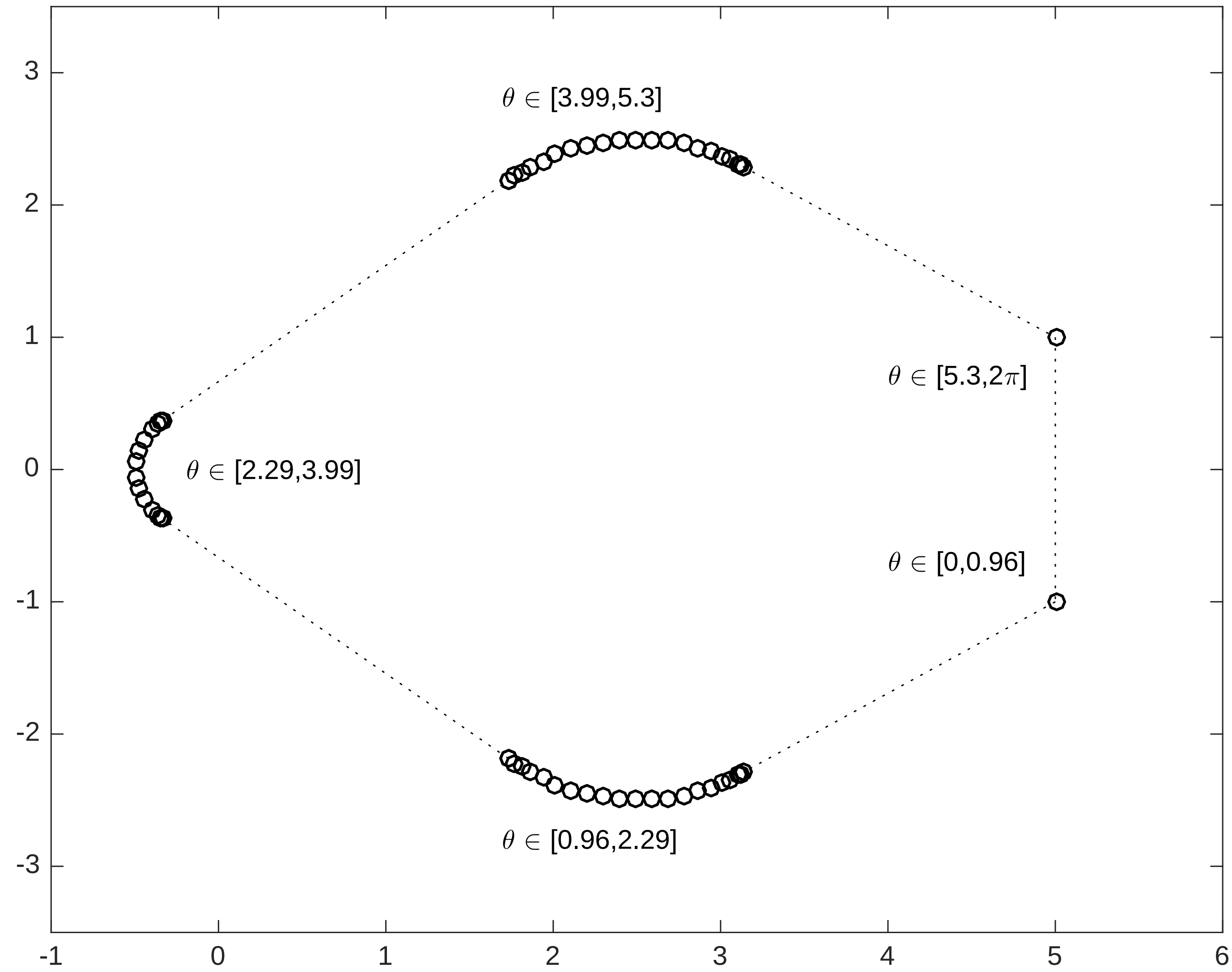

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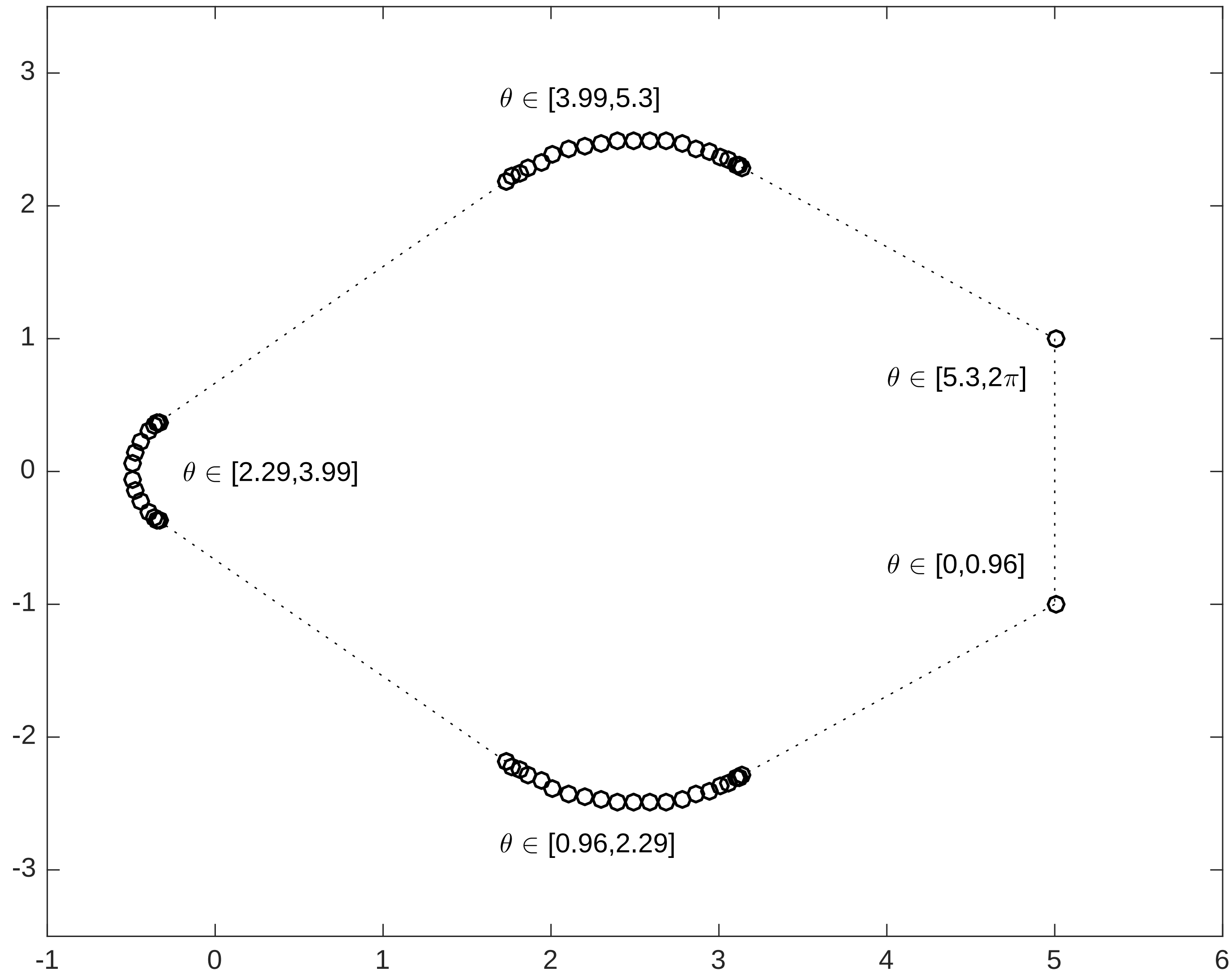
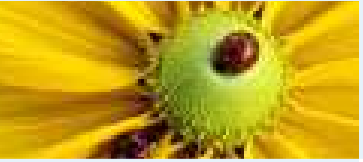


But how can we do this accurately, automatically and efficiently?



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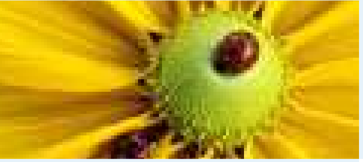

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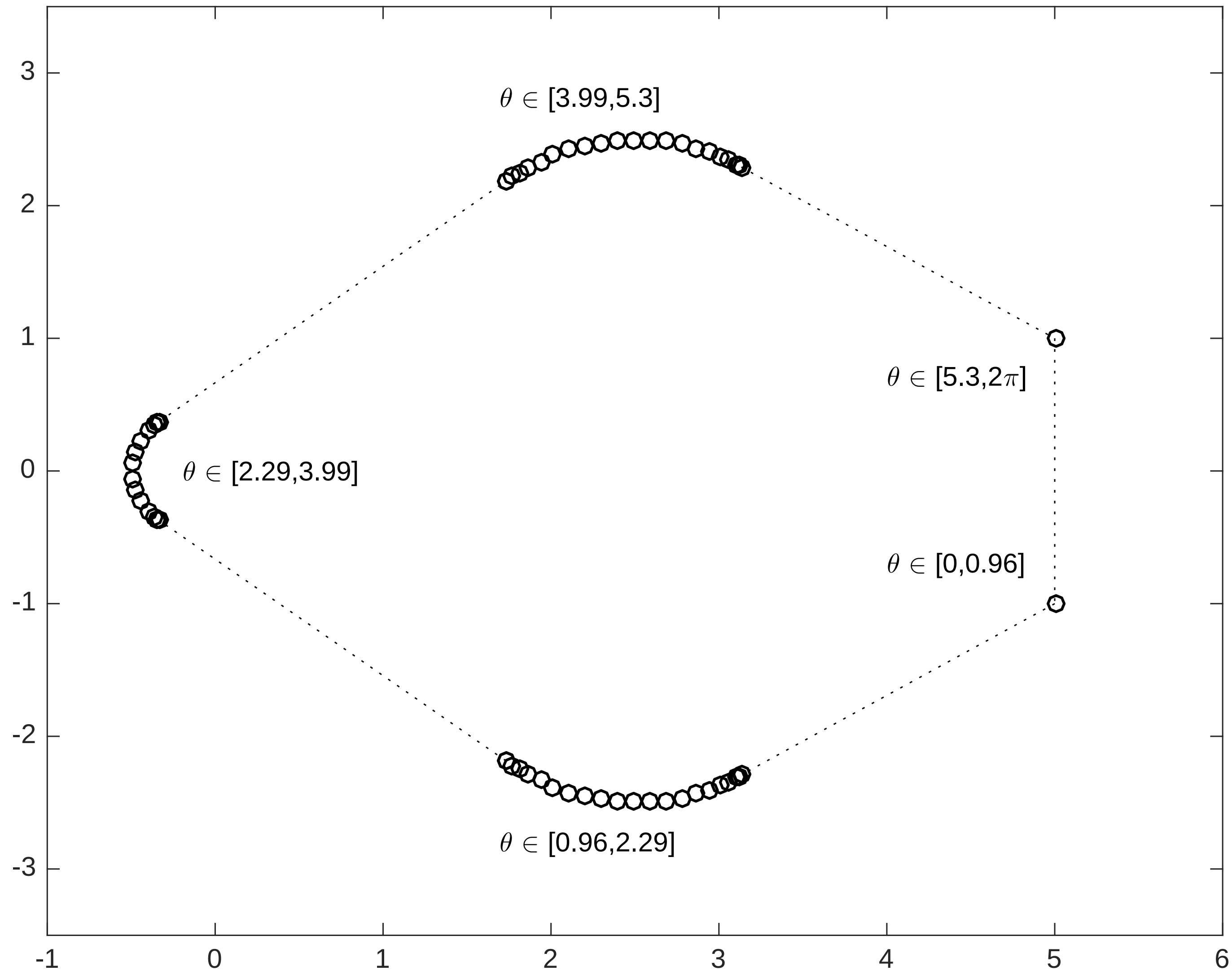
Applying Chebfun's **fov** to compute the boundary of $W(A)$ for the previous example...



The small circles are the interpolation points generated by Chebfun.

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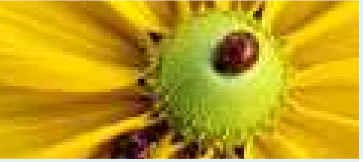

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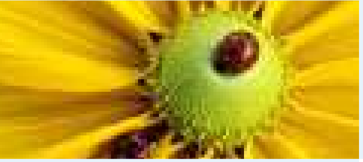

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- Semialgebraic (its graph is a finite union of sets, each of which is defined by a finite system of polynomial inequalities)

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The main cost is the construction of the chebfun defining the field of values.

Nonsmooth Optimization of the Crouzeix Ratio

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In all of these cases the gradient of f is not defined.
But in practice, none of these cases ever occur, except the first one *in the limit*.

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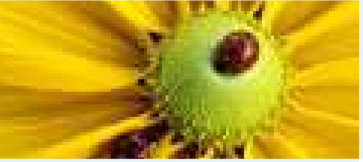
Remarkably, this property seems to extend to nonsmooth functions too, with a linear rate of local convergence, although the convergence theory is extremely limited (Lewis and Overton, 2013). It builds a very ill conditioned "Hessian" approximation, with "infinitely large" curvature in some directions and finite curvature in other directions.



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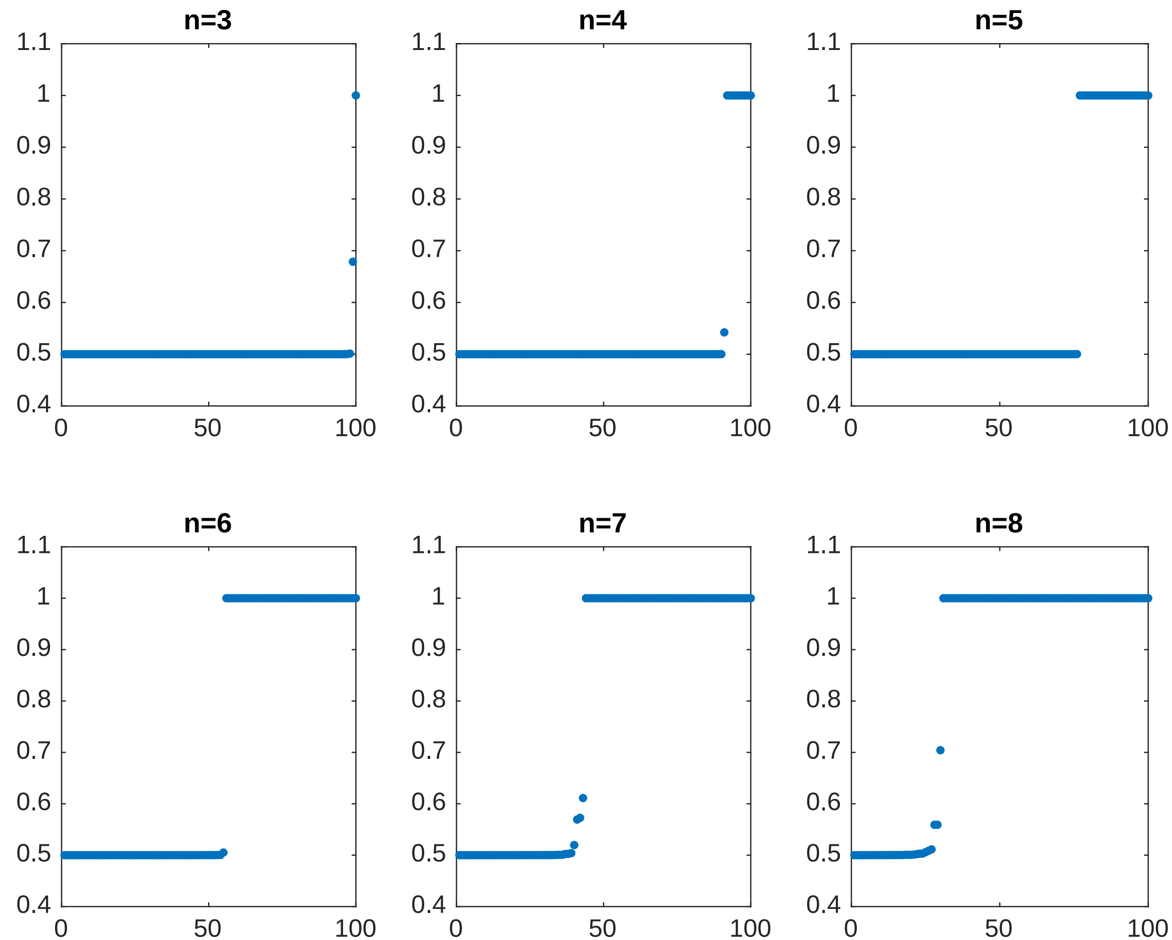
We have also obtained similar results using Gradient Sampling (Burke, Lewis and Overton, 2005; Kiwiel 2007) instead of BFGS. This method has a very satisfactory convergence theory, but it is much slower.

Optimizing over A (order n) and p (deg $\leq n-1$)



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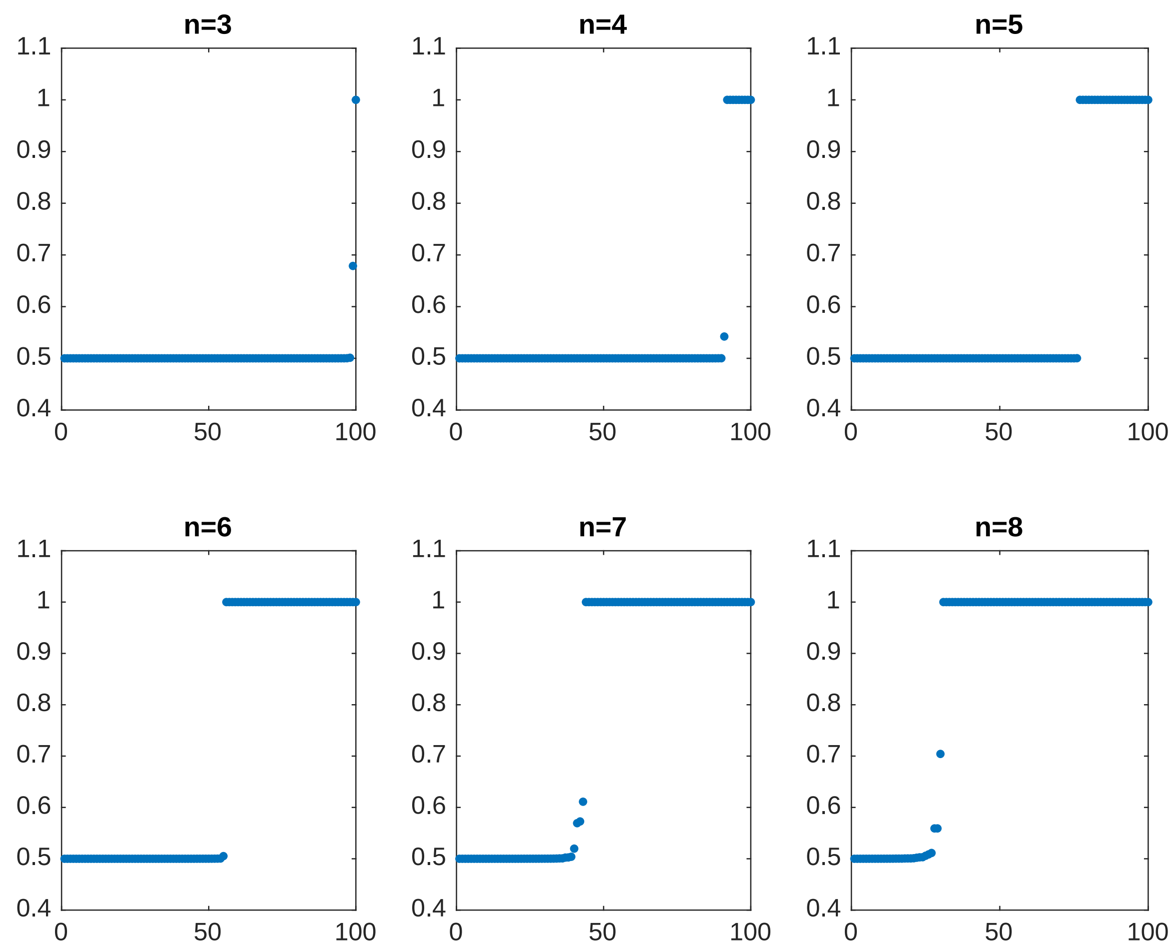
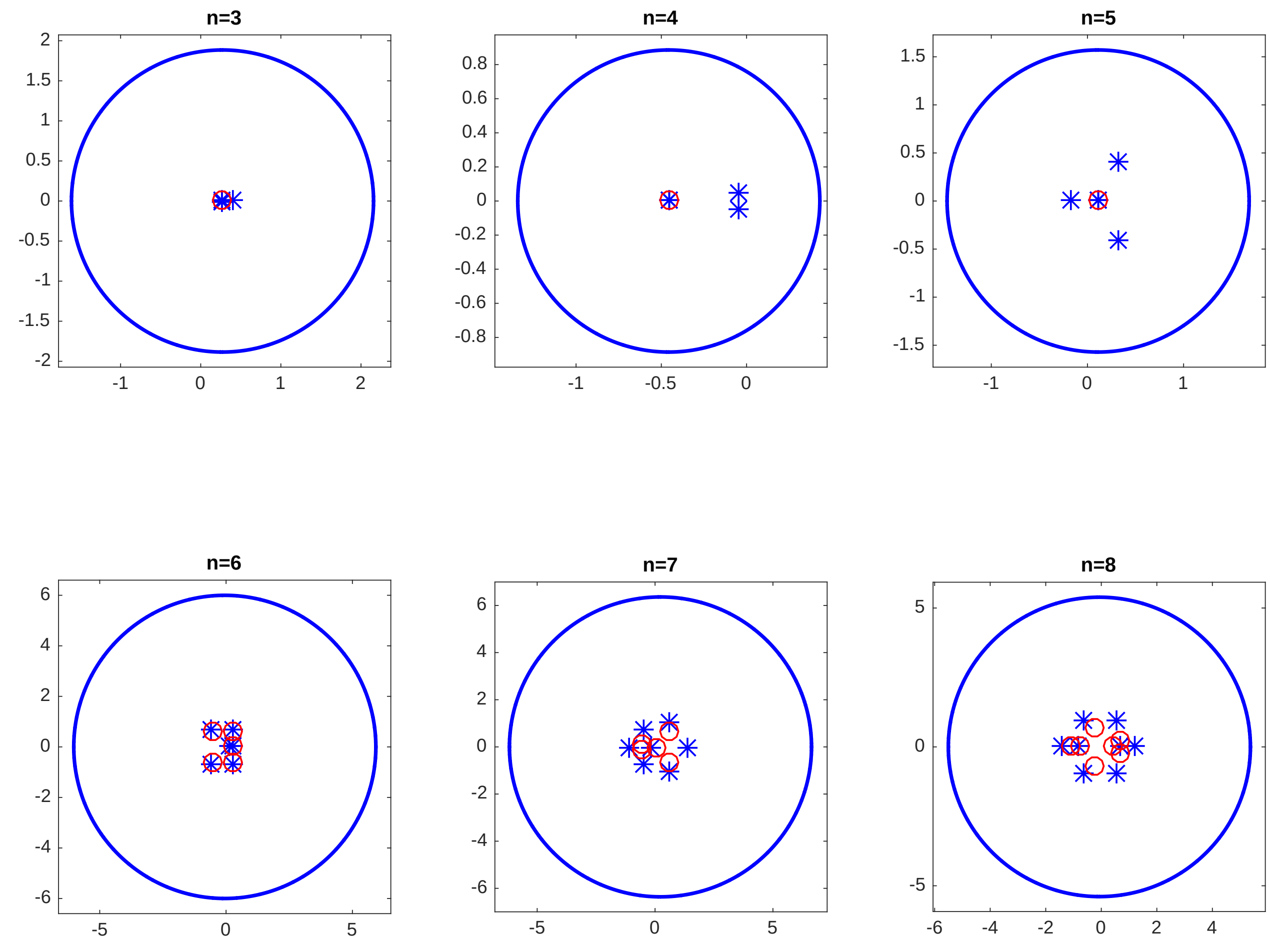


Sorted final values of the Crouzeix ratio f found starting from 100 randomly generated initial points.

Suggests that only locally optimal values of f are 0.5 and 1.

Final Fields of Values for Lowest Computed f



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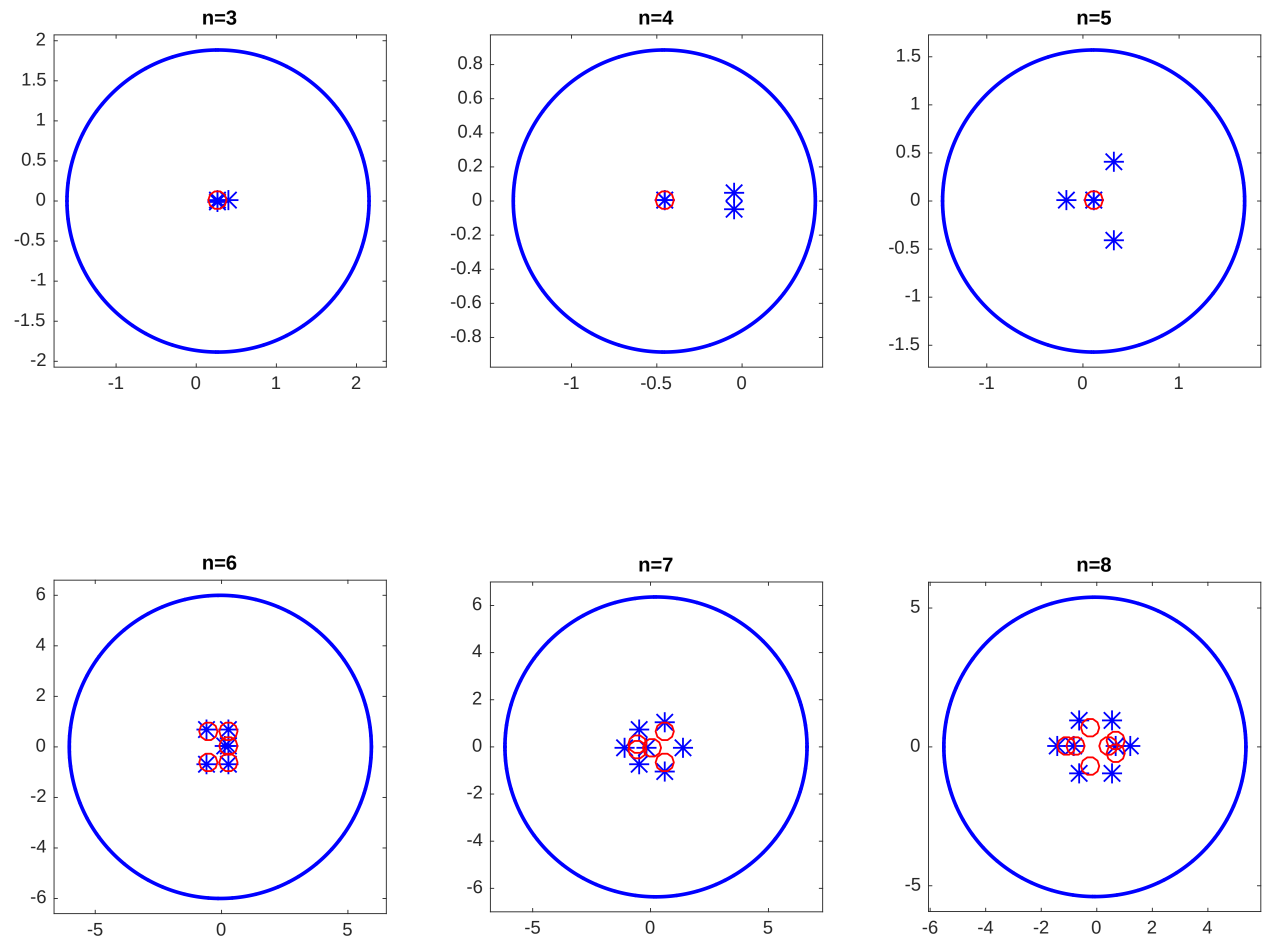

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$n = 3, 4, 5$: two eigenvalues of A and one root of p nearly coincident

Optimizing over both p and A: Final $f(p, A)$

n	f
3	0.500000000000000
4	0.500000000000000
5	0.500000000000014
6	0.500000017156953
7	0.500000746246673
8	0.500000206563813

f is the lowest value $f(p, A)$ found over 100 runs

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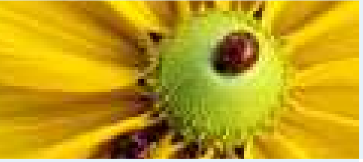
for which $W(A)$ is the unit disk.



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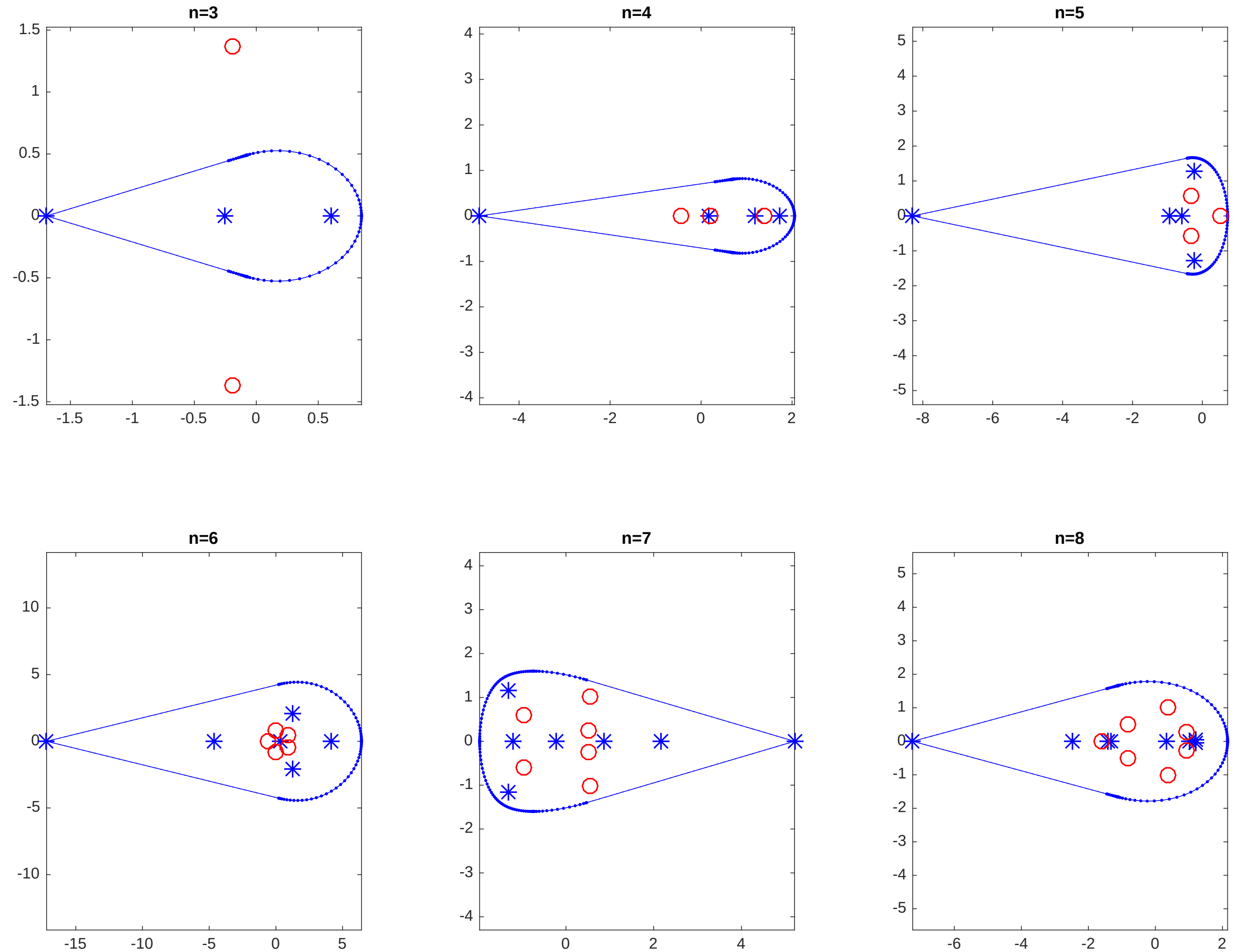
f is *nonsmooth* at these pairs (p, A) because $|p|$ is constant on the boundary of $W(A)$.

Final Fields of Values for f Closest to 1



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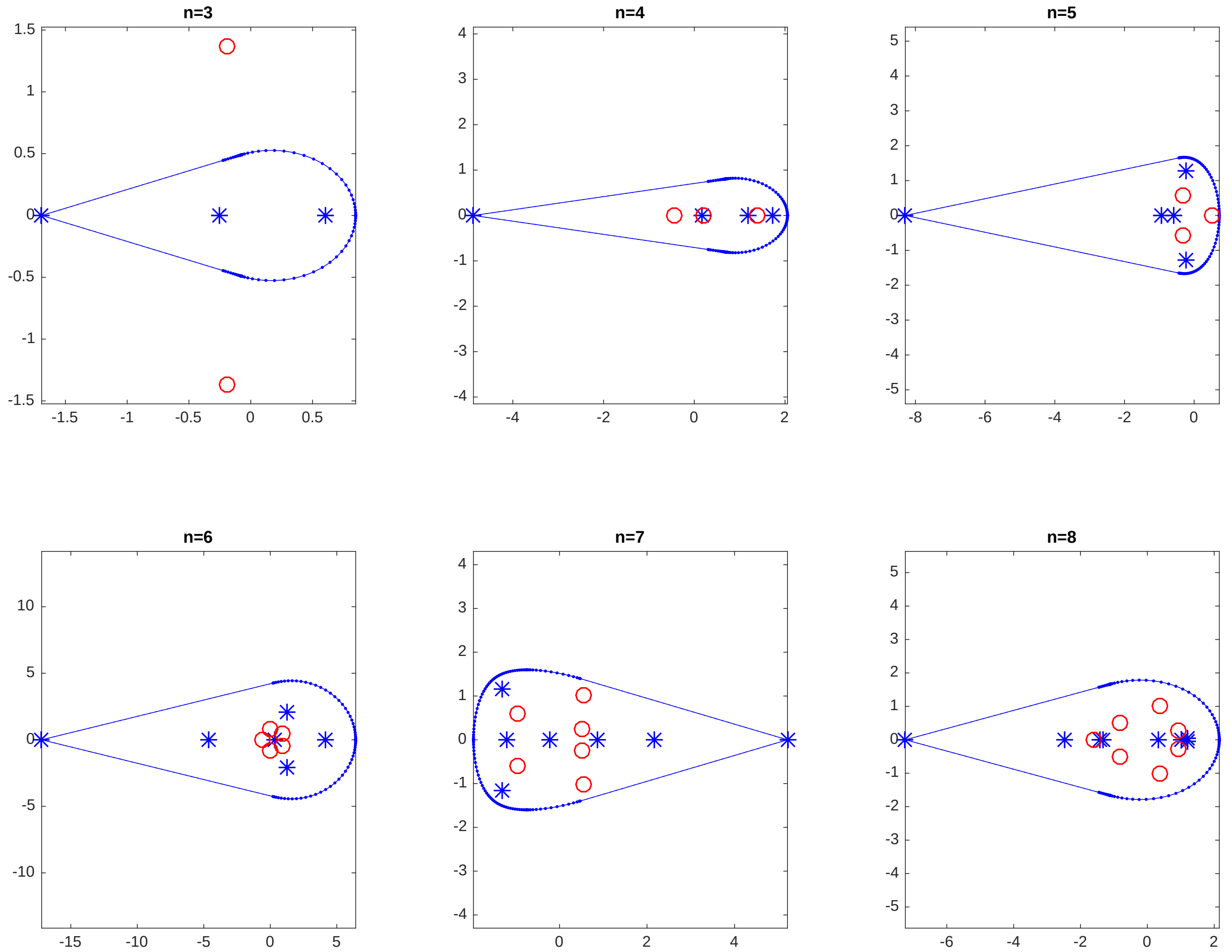

Ice cream cone shape:

exactly one eigenvalue at a vertex of the field of values

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so

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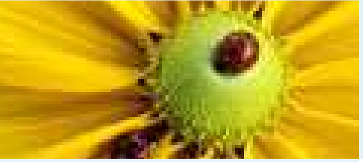

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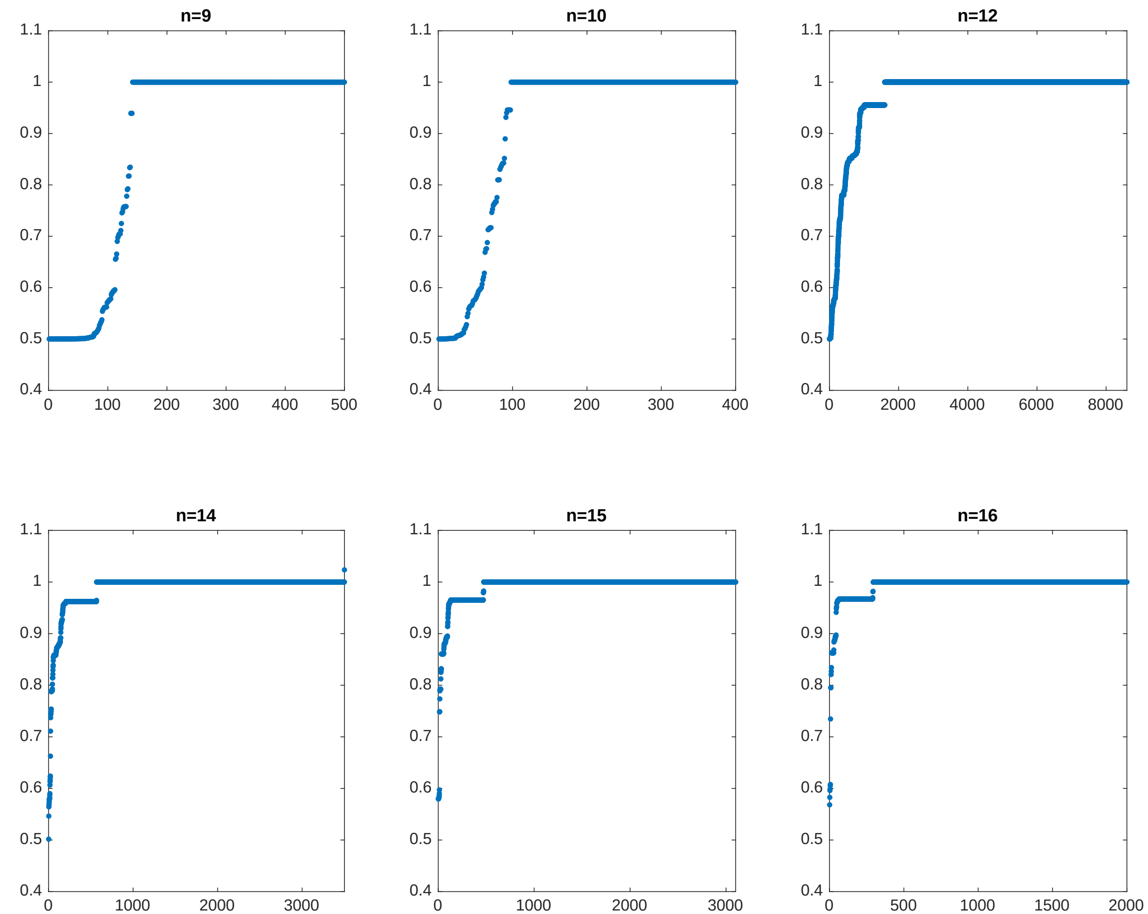
As n increases, ice cream cone stationary points become increasingly common and it becomes very difficult to reduce f below 1.

Results for Larger Dimension n and Degree $n-1$



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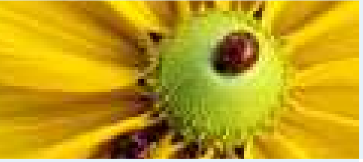

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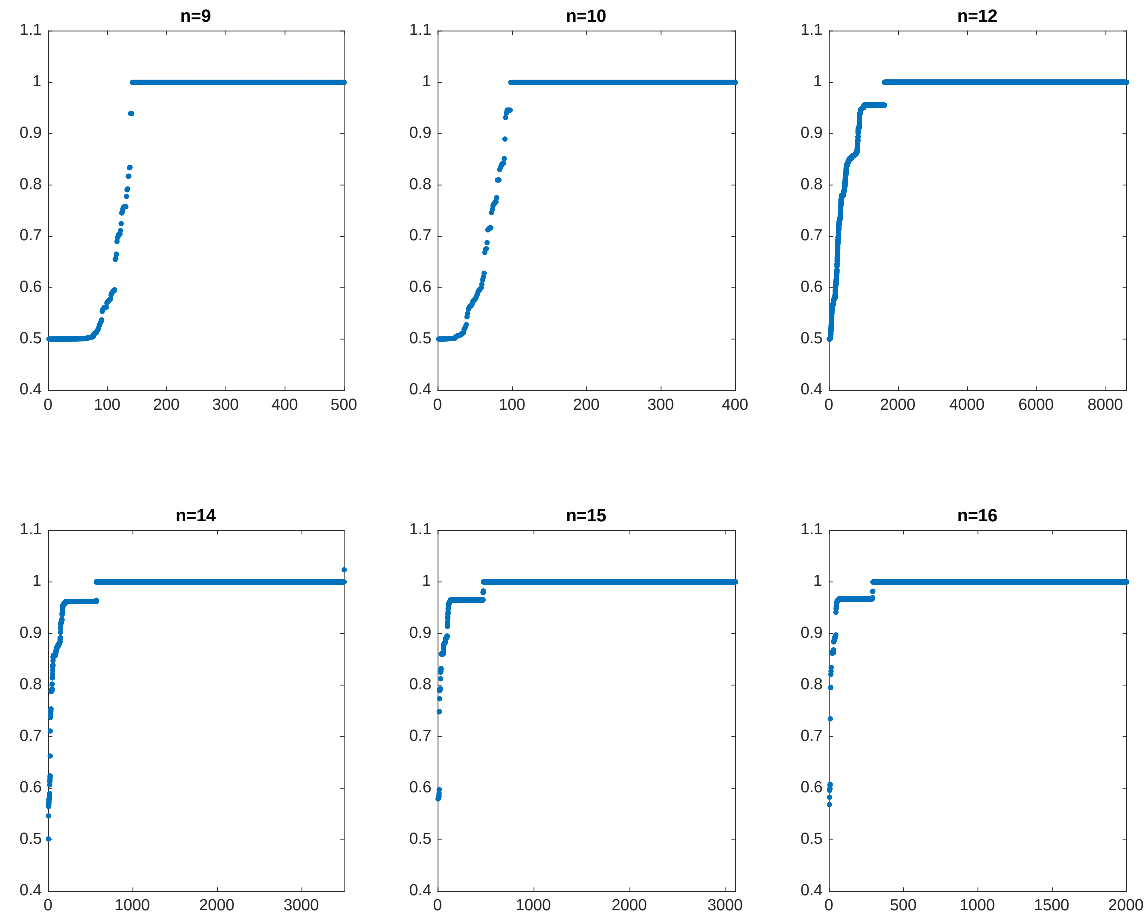

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There **are** other locally optimal values of f between 0.5 and 1 !

Nonsmooth Analysis of the Crouzeix Ratio

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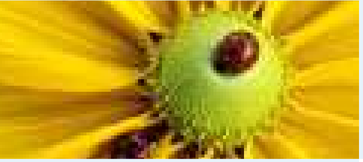

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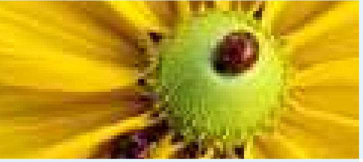

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Clarke stationarity is a *necessary condition* for local or global optimality.

The Gradient or Subgradients of the Crouzeix Ratio

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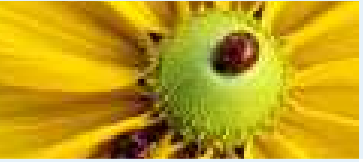

The Gradient or Subgradients of the Crouzeix Ratio

For the numerator, we need the variational properties of

$$\max_{\theta\in[0,2\pi]} |p(z_\theta)| \quad \text{where} \quad z_\theta = v_\theta^* A v_\theta.$$

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If the max of $|p(z_\theta)|$ is attained by a unique point $\hat{\theta}$, then all these are evaluated at $\hat{\theta}$ and combined with the gradient of $|\cdot|$ to obtain the gradient of the numerator.

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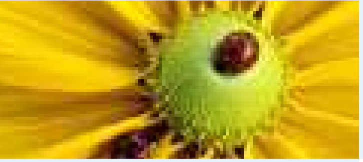

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Finally, use the quotient rule.

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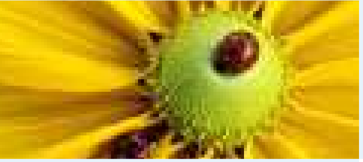

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- All continuously differentiable functions are regular
- Nonsmooth concave functions, e.g. $h(x) = -|x|$, are not regular.

Simplest Case where Crouzeix Ratio is Nonsmooth

Optimize over complex monic linear polynomials $p(\zeta) \equiv c + \zeta$ and complex matrices with order $n = 2$. Let $f(p, A) \equiv f(c, A)$, where now $f : \mathbb{C} \times \mathbb{C}^{2\times 2} \to \mathbb{R}$.

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Let $\hat{c} = 0$ ($\hat{p}(\zeta) = \zeta$) and $\hat{A} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, so $W(\hat{A}) = \mathcal{D}$, the unit disk, and hence $|p(\zeta)|$ is maximized everywhere on the unit circle, with f nonsmooth at $(\hat{c}, \hat{A})$ and $f(\hat{c}, \hat{A}) = 1/2$.

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Theorem 3. The Crouzeix ratio f is regular at $(\hat{c}, \hat{A})$, with

$$\partial f(\hat{c}, \hat{A}) = \mathrm{conv}_{\theta\in[0,2\pi)} \left\{ \left(\frac{1}{2} e^{-i\theta}, \frac{1}{4} \begin{bmatrix} e^{-i\theta} & 0 \\ e^{-2i\theta} & e^{-i\theta} \end{bmatrix} \right) \right\}$$

Corollary.

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Actually, we knew this must be true as Crouzeix's conjecture is known to hold for $n = 2$, and hence $(\hat{c}, \hat{A})$ is a global minimizer of $f(\cdot, \cdot)$, but we can extend the result to larger values of m, n, for which we don't know whether the conjecture holds.

The General Case

Optimize over complex polynomials $p(\zeta) \equiv c_0 + \cdots + c_m\zeta^m$ and complex matrices with order n. Let $f(p, A) \equiv f(c, A)$, where $f : \mathbb{C}^{m+1} \times \mathbb{C}^{n\times n} \to \mathbb{R}$.

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Theorem 4. The Crouzeix ratio on $(c, A) \in \mathbb{C}^{m+1} \times \mathbb{C}^{n\times n}$ is regular at $(\hat{c}, \hat{A})$ with

$$\partial f(\hat{c}, \hat{A}) = \mathrm{conv}_{\theta\in[0,2\pi)}\Big\{\big(y_\theta, Y_\theta\big)\Big\}$$

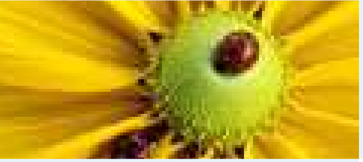
where

$$y_\theta = \frac{1}{2}\left[z^m, z^{m-1}, \ldots, z, 0\right]^T$$

and Y_θ $n\times n$ matrix

$$Y_\theta = \frac{1}{4}\begin{bmatrix} z & 0 & \sqrt{2}z^{-1} & \sqrt{2}z^{-2} & \cdots & \sqrt{2}z^{3-n} & z^{2-n} \\ \sqrt{2}z^2 & 2z & 0 & 2z^{-1} & \cdots & 2z^{4-n} & \sqrt{2}z^{3-n} \\ \vdots & & & & & & \vdots \\ \sqrt{2}z^{n-2} & 2z^{n-3} & 2z^{n-4} & 2z^{n-5} & \cdots & 0 & \sqrt{2}z \\ \sqrt{2}z^{n-1} & 2z^{n-2} & 2z^{n-3} & 2z^{n-4} & \cdots & 2z & 0 \\ z^n & \sqrt{2}z^{n-1} & \sqrt{2}z^{n-2} & \sqrt{2}z^{n-3} & \cdots & \sqrt{2}z^2 & z \end{bmatrix}$$

with $z = e^{-i\theta}$.



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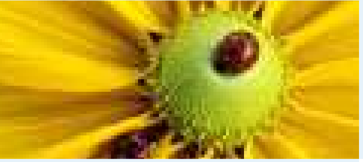

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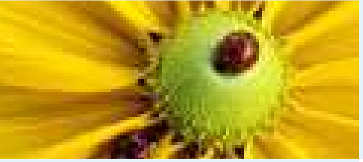
Is the Crouzeix Ratio Globally Clarke Regular?



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No. Let $\tilde{p}(\zeta) = \zeta$ and

$$\tilde{A} = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

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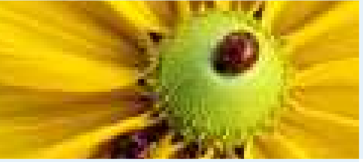

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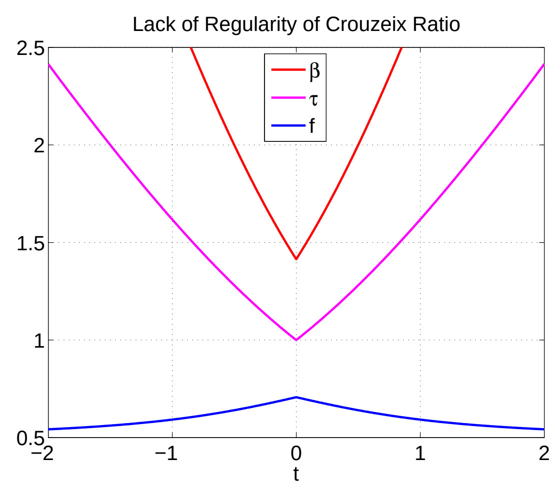
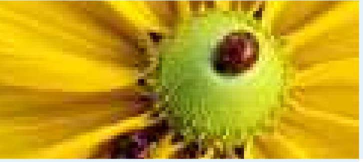


Plot of the denominator β, the numerator τ and the Crouzeix ratio f evaluated at $(\tilde{p}, \tilde{A} + t\tilde{A}^2)$, $t \in [-2, 2]$.

Concluding Remarks



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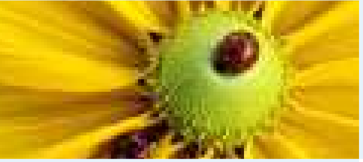
Minimizing the Crouzeix ratio f over p and A, BFGS almost always converged either to *nonsmooth stationary* values of 0.5 associated with the Crabb-Choi-Crouzeix matrix (with field of values a disk), or *smooth stationary* values of 1 (with "ice cream cone" fields of values).



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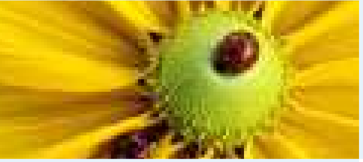

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The results strongly support Crouzeix's conjecture: the globally minimal value of the Crouzeix ratio $f(p, A)$ is 0.5.

Our Papers

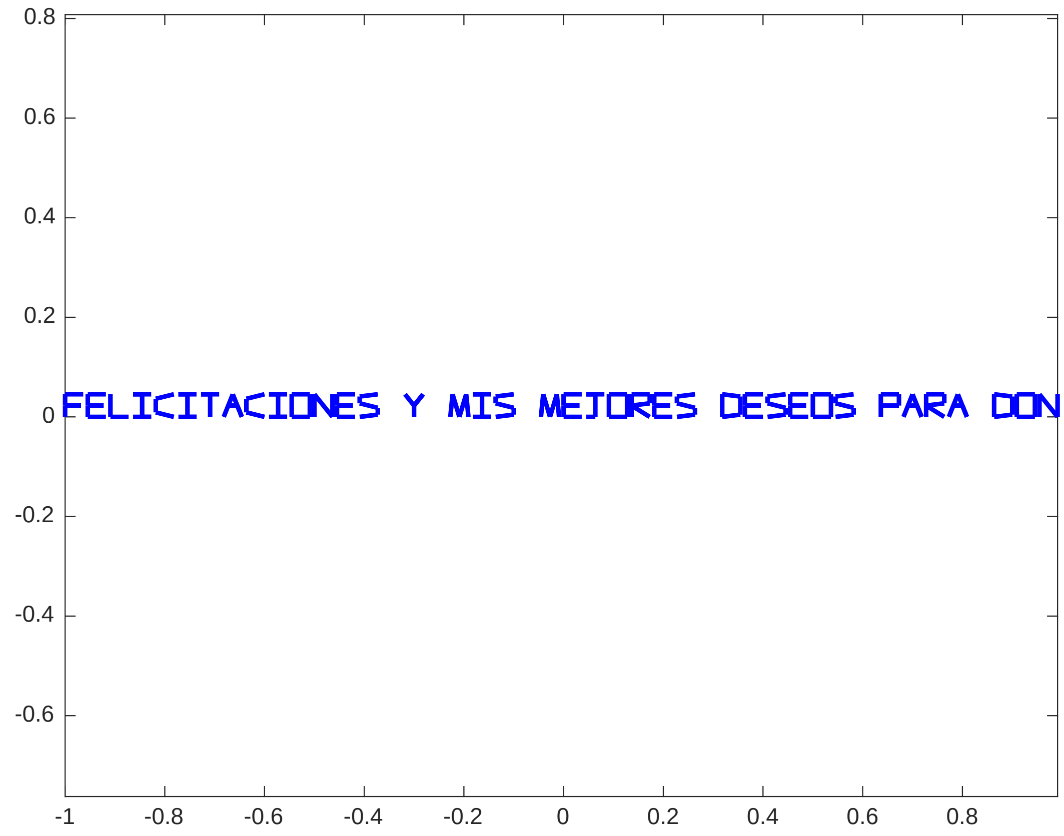
A. Greenbaum and M.L. Overton
Investigation of Crouzeix's Conjecture via Nonsmooth Optimization
Linear Alg. Appl., 2017

A. Greenbaum, A.S. Lewis and M.L. Overton
Variational Analysis of the Crouzeix Ratio
Math. Programming, 2016

A.S. Lewis and M.L. Overton
Nonsmooth Optimization via Quasi-Newton Methods
Math. Programming, 2013

A Chebfun Message to Don

```
% define and plot a chebfun with 338 pieces
s=scribble('Felicitaciones y mis mejores deseos para Don');
plot(s,'b','LineWidth',2), axis equal
```



Or, More Circularly

```
plot(exp(3i*s),'m','LineWidth',2), axis equal
```