A Variational Approach of Waveform Design In Tomographic Imaging Systems

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Itinerary

1. Inverse Problems, Wave Scattering, and Imaging

2. Waveform Design for Synthetic-Aperture Imaging Systems

3. Conclusions and Remarks

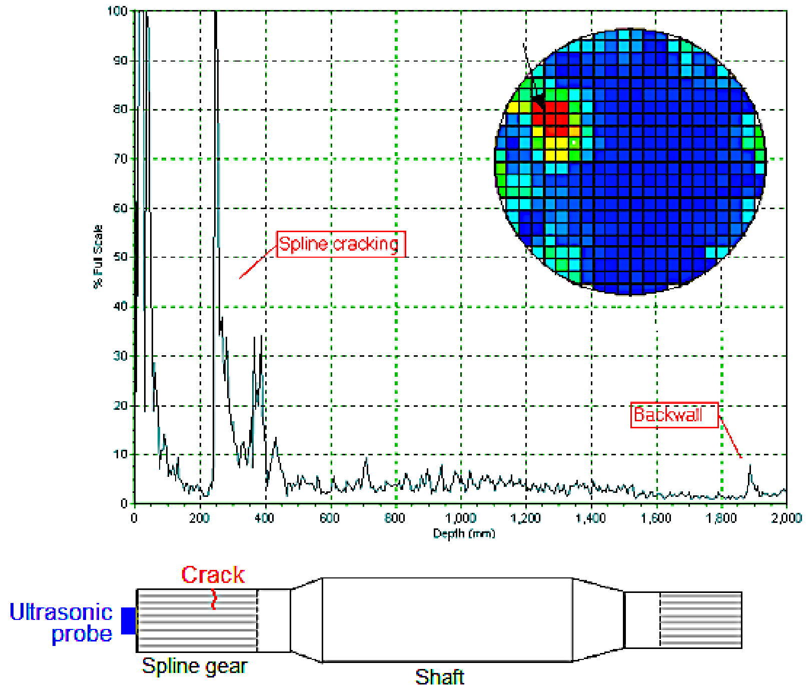
Objective: Developing a theory for optimizing waveform design on imaging through a dispersive medium

In collaboration with

- **Margaret Cheney**, Colorado State University (formerly RPI)
- **Trond Varslot**, Thermo Fisher Scientific & Norwegian University of Science and Technology

Imaging Inverse Problems

Definition 1 (Inverse Problem). *The mathematical problem of retrieving information of unknown quantities by indirect observations.*



Definition 2 (Image). *A* ***map*** *of the estimated* ***loci of points*** *at which* ***wave speed*** *has singularities: material discontinuities.*

Wave-Based Systems

- **Tomographic Reconstruction**: finding a density distribution $T \in \Omega$ from all its line integrals (projections) $g = \mathcal{R}T$, where $\mathcal{R}$ denotes the **Radon Transform**:

$$\mathcal{R}T(s, \boldsymbol{\theta}) := \int_{\ell(s,\boldsymbol{\theta})\cap\Omega} T(\boldsymbol{x})d\sigma(\boldsymbol{x}).$$

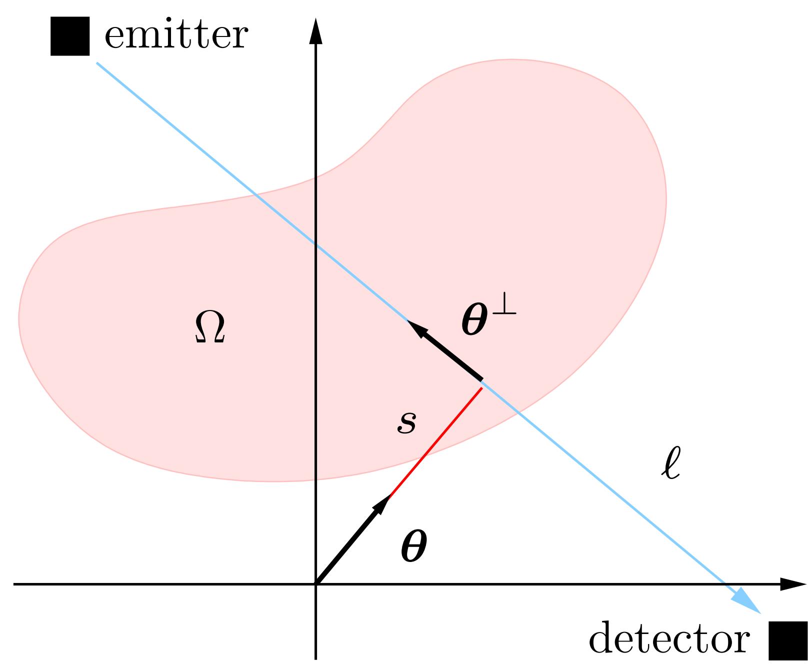
- Analytically tomographic reconstruction is represented by the inversion formula

$$T = (2\pi)^{-1}\mathcal{R}^*\Lambda g,$$

where the backprojection operator $\mathcal{R}^*$ is the adjoint to $\mathcal{R}$,

$$\mathcal{R}^*g(\boldsymbol{x}) := \int_0^\pi g(\boldsymbol{\theta}, \boldsymbol{x}\cdot\boldsymbol{\theta})d\boldsymbol{\theta}.$$

Formally, Λ is the square root of the 1D Laplacian.



Wave-Based Systems

- Wavefield **forward model** for acoustic and electromagnetic **imaging systems** is a non-constant coefficient scalar **wave equation** (linear)

$$\Delta u(t, \boldsymbol{x}) - \partial_t^2 \int_0^\infty c_0^{-2} n^2(\tau, \boldsymbol{x}) u(t - \tau, \boldsymbol{x}) d\tau = -j.$$

- Scattered wavefields $u_S(t, \boldsymbol{x}, \boldsymbol{y})$ propagate singularities of refractive index n, subject to a source $j(t, \boldsymbol{x}, \boldsymbol{y})$, initial and boundary conditions.

- **Dispersive medium** means that phase velocity v or refractive index n depend on frequency ω
$$\frac{1}{v^2(\omega)} = \frac{n^2(\omega)}{c_0^2}.$$

- For a broadband incident signals, all material are *dispersive* to some extent causing wave attenuation (distorsion).

Fourier Integral Operators (FIO) Models

Imaging modalities: Emission Tomography (PET/SPECT), Doppler Ultrasound, Seismic Imaging, Electron Tomography.

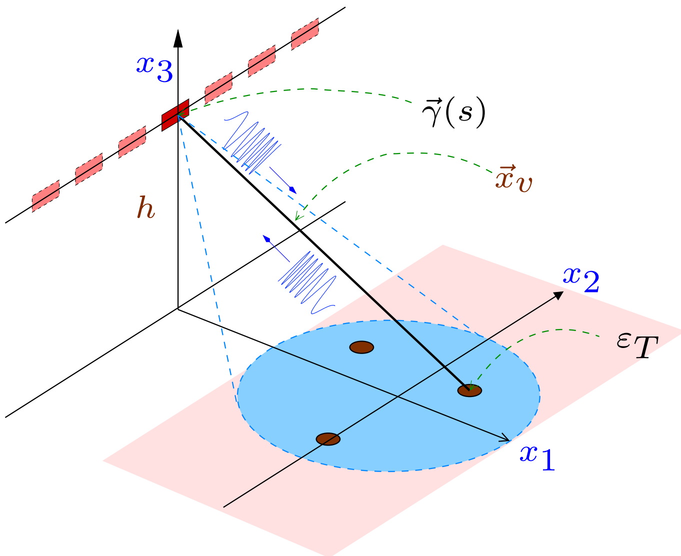
- The noise-free forward model (echo or transmitted field) u_S, under the Born approximation, adopts the form of a Fourier Integral Operator (FIO):

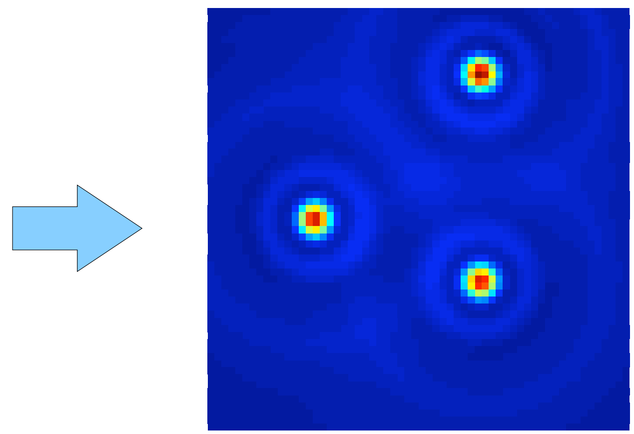
$$u_S(\boldsymbol{y}) = \int e^{-i\Phi(\omega,\boldsymbol{x},\boldsymbol{y})} A(\omega,\boldsymbol{z},\boldsymbol{y}) T(\boldsymbol{z}) \mathrm{d}\omega \mathrm{d}\boldsymbol{z}$$

Measurements u are **weighted projections** of T over **manifolds** (lines, circles, spheres, etc.) defined by the phase function $\Phi(\omega,\boldsymbol{x},\boldsymbol{y})$ and $A(\omega,\boldsymbol{z},\boldsymbol{y})$ is the **amplitude function**.

- The images can be “blurred” due to the term A, that is a distinctive attribute of the system and sensing parameters.

Synthetic-Aperture Radar (SAR) Imaging System, '60s





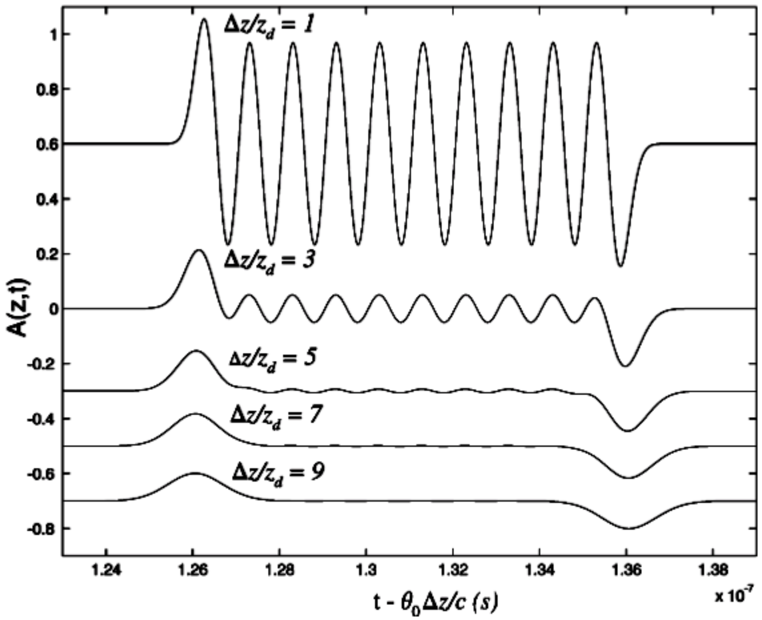
SAR is an active *remote sensing system*: monitoring ocean waves, soil humidity, forest ecology, planets' cartography, etc.

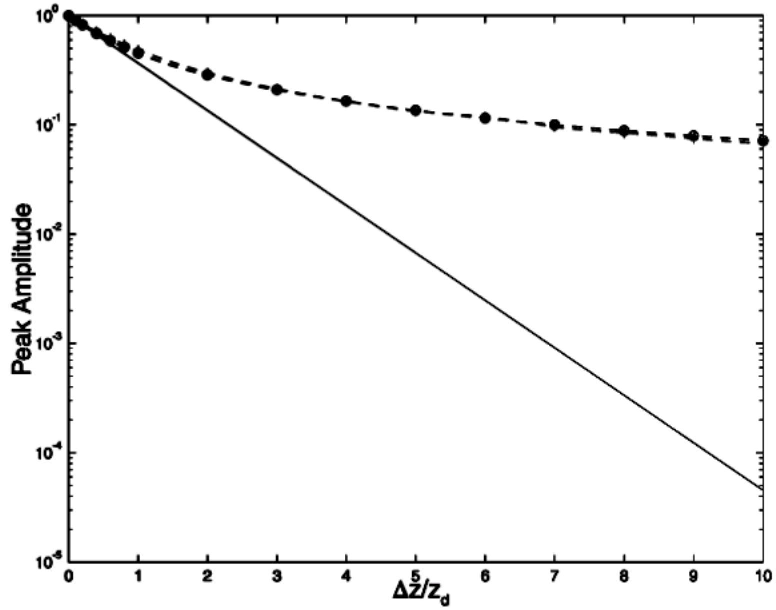
1. A *moving antenna* transmits and collects electromagnetic pulses.

2. The Antenna's Effective-Aperture is *sythesized*.

3. *Signal Processes* for Imaging.

Motivation: Wave Propagation in Dispersive Media





- **Peak amplitude** in the Brilluoin precursor **decays** algebraically (Oughstun, 2005).
- $u(t,x) \approx u_B(t,x) + u_c(t,x)$.
- **Can we use PRECURSORS for remote sensing or ultrasound?**

Brillouin precursor

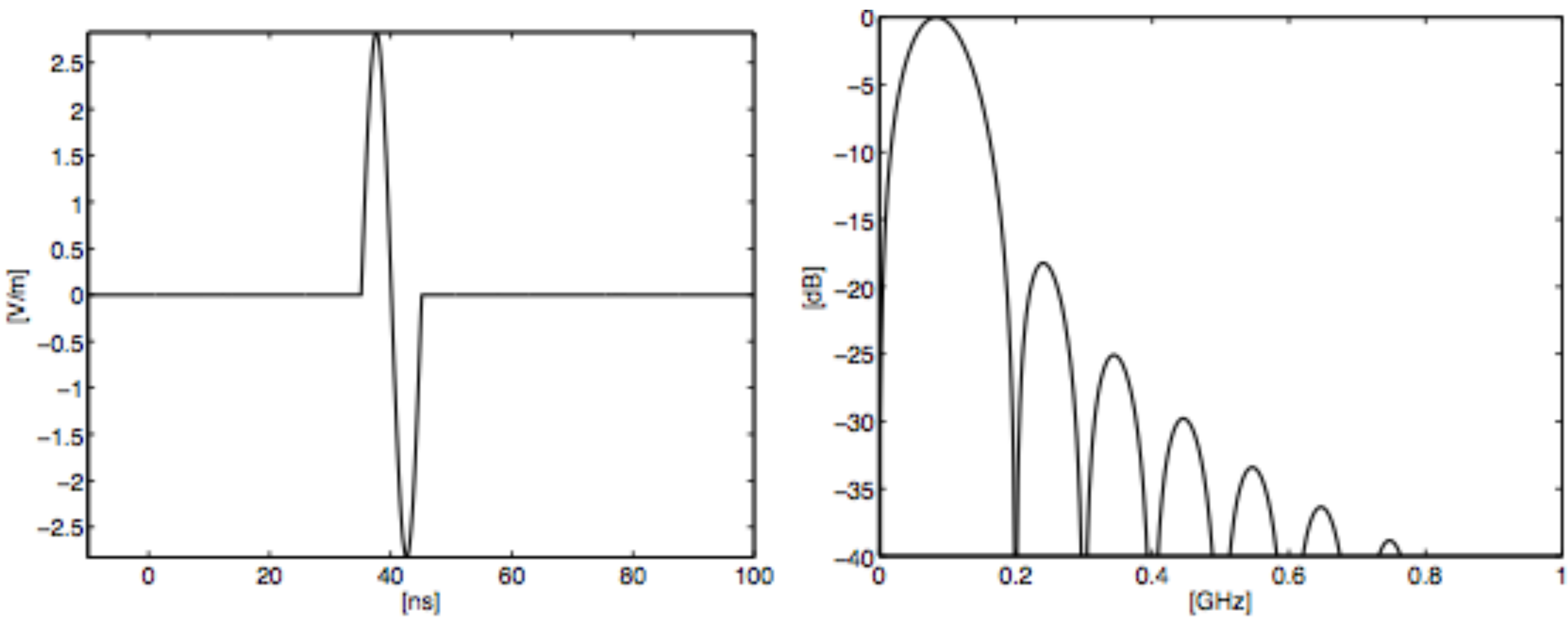
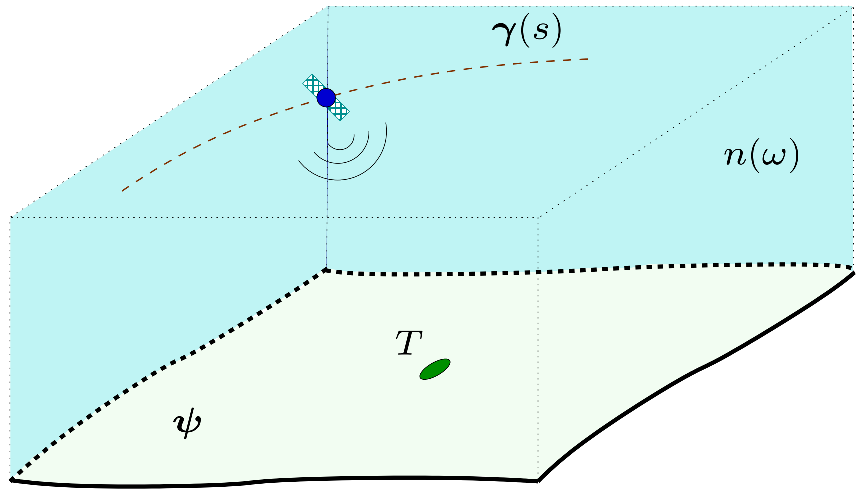


Figure 1: Ultra-short one-cycle sinusoid and its spectrum develops a transient waveform (precursor) with an algebraically amplitude decay rather than exponential decay.

(Electromagnetic) Scattering Theory



- Dispersor T (medium's inhomogeneity) means permittivity's perturbation

$$\varepsilon(s,\boldsymbol{x}) = \varepsilon_0\varepsilon_r(s,\boldsymbol{x}) + \overbrace{\varepsilon_0\varepsilon_T(x_1,x_2)\delta_0(x_3 - x_3^0)}^{\text{perturbative term}} \tag{a}$$

$$\frac{1}{v^2(\omega,\boldsymbol{x})} = \frac{1}{c_0^2}\left\{\varepsilon_r(\omega,\boldsymbol{x}) + \varepsilon_T(x_1,x_2)\delta_0(x_3 - x_3^0)\right\} = \frac{\varepsilon_r(\omega,\boldsymbol{x})}{c_0^2} + T(\boldsymbol{x})\delta_0, \tag{b}$$

$$\Delta u - \partial_t^2(\underbrace{c_0^{-2}\varepsilon_r}_{\text{medium}} *_t u) = \underbrace{(c_0^{-2}\varepsilon_T\delta_0)}_{\text{dispersor}}\,\partial_t^2 u - \underbrace{(\mu_0\partial_t\boldsymbol{J})_{x_i}}_{\text{source}} \tag{c}$$

Scattering Theory and the Inverse Problem

Theorem 1 (Direct Problem). *Let* $\boldsymbol{y} \in \mathbb{R}^2$ *such that* $(\boldsymbol{y}, \psi(\boldsymbol{y}))$ *is a point on the surface* Φ *and let* $\mathrm{d}\boldsymbol{y}$ *be the 2D surface measure. Under the single-scattering (Born) approximation, the scattered scalar wavefield* u^{sc} *is*

$$u_B^{sc}(\boldsymbol{\gamma}(s), t, s) = \int \frac{e^{-i\omega(t-2n(\omega)|\boldsymbol{z}-\boldsymbol{\gamma}(s)|/c_0)}}{(4\pi)^2|\boldsymbol{z}-\boldsymbol{\gamma}(s)|^2}\omega^2 P(\omega) T(\boldsymbol{z}) \boldsymbol{\Psi}(\boldsymbol{y}) \mathrm{d}\omega \mathrm{d}\boldsymbol{z}.$$

$P(\omega)$ *incident pulse (FT),* $T(\boldsymbol{z})$ **reflectivity function** *or target, and* $\boldsymbol{\Psi}$ *is the curvature scattering area on nonplanar topography.*

- Adding (thermal) noise $d(t, s) = u_B^{sc}(\boldsymbol{\gamma}(s), t, s) + \eta(t)$ we get our "data".

Proposition 1 (Inverse Problem). *To form an image* $I(\boldsymbol{z})$ *(i.e., an estimate of* $T(\boldsymbol{z})$*), we apply to the noisy data* $d(t, s)$ *an operator* B_Q *of the form*

$$I(\boldsymbol{z}) = \int e^{i\omega'\left(t-2n_R(\omega')|\boldsymbol{z}-\boldsymbol{\gamma}(s)|/c_0\right)} Q(\omega', \boldsymbol{z}, s) d(t, s) \mathrm{d}\omega' \mathrm{d}s \mathrm{d}t,$$

where the (regularization) filter Q *must be determined.*

Scattering Theory and the Inverse Problem

Proposition 2 (Inverse Problem). *Estimate* $T(\boldsymbol{z})$ *from* $d(t,s)$ *by means of backprojection (matched filter, migration)* $B_Q[d] \mapsto I(\boldsymbol{z})$.

- Key property of B_Q: complex conjugate phase of $F[T] \mapsto d$.
- Changing variables (Stolt) $(\omega, s) \to \boldsymbol{\xi} \in \Omega$ and substituting $d = F[T]$ into $I = B_Q[d]$:

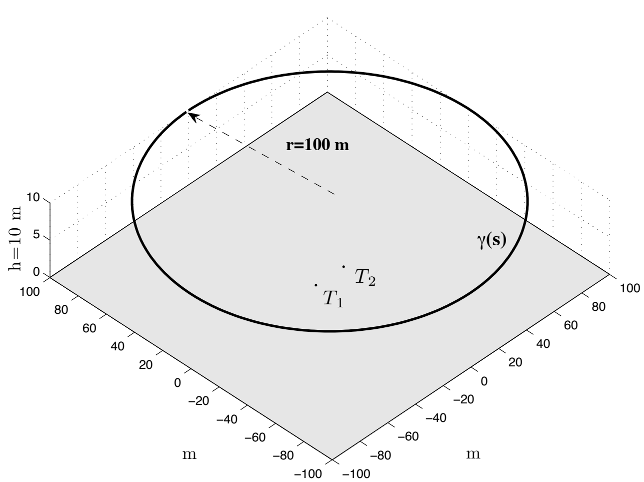
$$I(\boldsymbol{z}) = \int e^{i(\boldsymbol{z}-\boldsymbol{y})\cdot\boldsymbol{\xi}} Q(\boldsymbol{\xi}, \boldsymbol{z}) a(\boldsymbol{\xi}, \boldsymbol{y}) P(\boldsymbol{\xi}) T(\boldsymbol{y}) J(\boldsymbol{\xi}, \boldsymbol{z}, \boldsymbol{y}) \mathrm{d}\boldsymbol{\xi} \mathrm{d}\boldsymbol{y} + I_n(\boldsymbol{z}),$$

where $a = \omega^2 e^{-2\omega n_I(\omega)|\boldsymbol{\gamma}(s)-\boldsymbol{y}|/c_0}/(4\pi|\boldsymbol{\gamma}(s) - \boldsymbol{y}|)^2$ comprises geometrical factors.

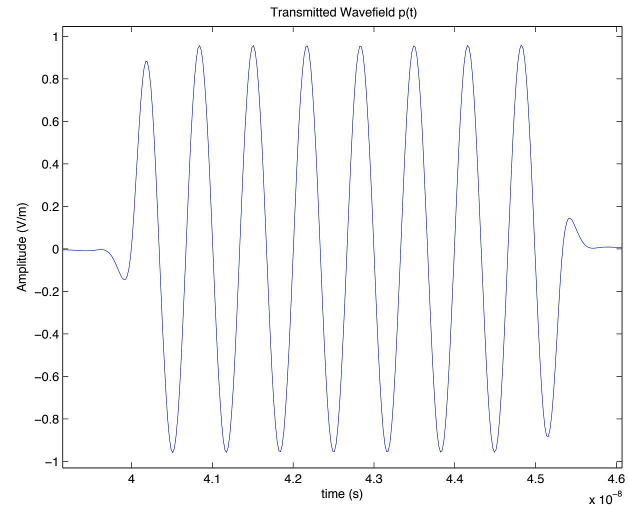
Definition 3 (Ideal Reflectivity). *(Sifting (sampling) Dirac's delta property.)*

$$I_\Omega(\boldsymbol{z}) = \int_{\boldsymbol{\xi}\in\Omega} e^{i(\boldsymbol{z}-\boldsymbol{y})\cdot\boldsymbol{\xi}} T(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{\xi}, \quad Q = (a\, P\, J)^{-1} \quad \textit{and} \quad \eta = 0.$$

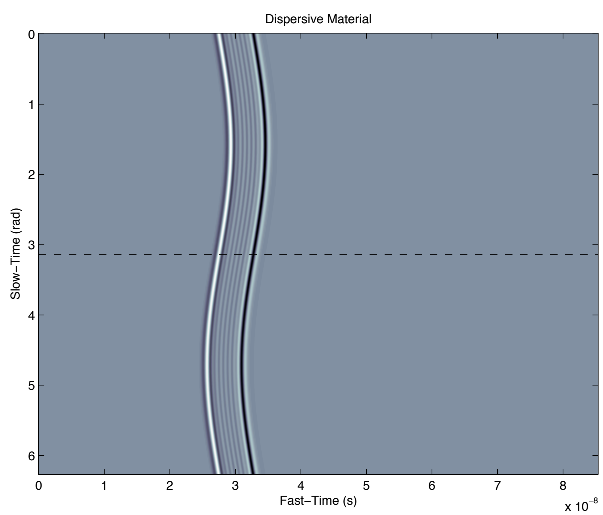
Dispersion and noiseless computational setup



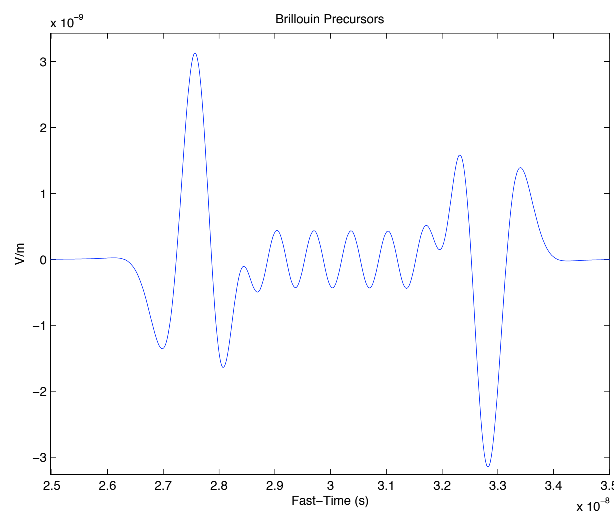
(b) Scenario Two Scatteres



(c) Incident 8-cycle sinusoid

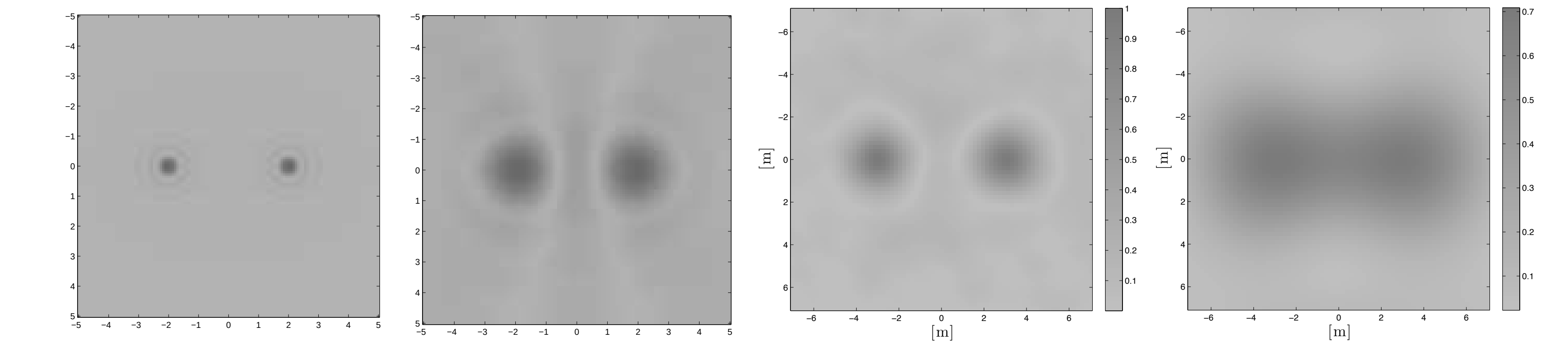


(d) Sinograms

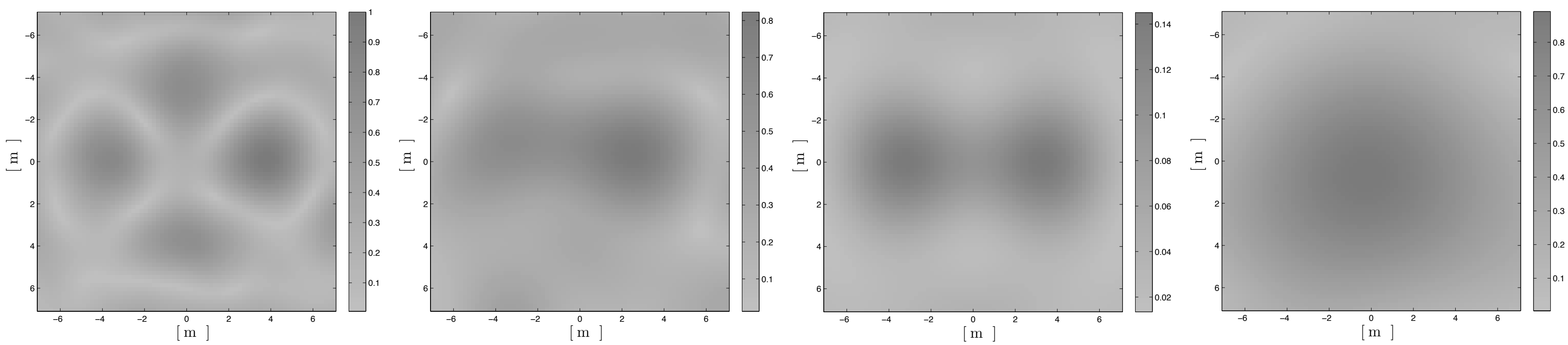


(e) Wave distorted (precursors)

Computational simulations



(f) ND (g) D (h) MN (i) MD



(j) MN (k) MD (l) HD (m) HN

Figure 2: Key: ND no dispersive, D dispersive, MN moderate noise, MD moderate dispersive, HD High Dispersive, HD high dispersive. Relative units.

Optimal Filter: Minimum Mean-Square-Error ◇

Definition 4 (Mean-Square-Error (MSE)). *A filtered-backprojection-type reconstruction method that minimizes the MSE:*

$$\Delta(Q,P) = \int \langle |I(\boldsymbol{z}) - I_{\Omega_{\boldsymbol{z}}}(\boldsymbol{z})|^2 \rangle \mathrm{d}\boldsymbol{z}$$

Theorem 2 (Optimal Reconstruction Filter). *The filter Q that is optimal, in the sense of minimizing the variance $\Delta(P,Q)$ in the leading order contributions of the image, is approximated by*

$$Q^{opt}(\omega, s, \boldsymbol{y}) = \frac{\overline{P(\omega,s)A(\omega,s,\boldsymbol{z})}}{|P(\omega,s)a(\omega,s,\boldsymbol{z})|^2 J(\omega,s,\boldsymbol{z}) + \frac{S_\eta(\omega,s)}{S_T(\omega,s,\boldsymbol{y})}},$$

where S_η/S_T is the (frequency-domain) noise-to-target ratio.

Optimal Filter: Minimum Mean-Square-Error ◇

Dem. 1 (Sketch). *Because the target and the noise are statistically independent and have zero mean, we can split up*

$$\Delta(P,Q) = \Delta_T(P,Q) + \Delta_\eta(P,Q).$$

Under Stolt change of variables and stationary phase method (variable reduction) we get

$$\Delta(P,Q) \approx \int |QA_P J - 1|^2 \frac{S_T}{J} \mathrm{d}\omega ds \mathrm{d}\boldsymbol{z} + \int |Q|^2 S_\eta \mathrm{d}\omega ds \mathrm{d}\boldsymbol{y}.$$

Finally, we apply the first variation

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}\rho}\right|_{\rho=0} \Delta(Q + \rho Q_\rho, P).$$

Details (Varslot, M, Cheney, 2010).

Optimal Waveform Spectrum Design

- We minimize the mean-square-error (MSE) of the leading order contributions of the resulting image I w.r.t. $I_{\Omega_{\boldsymbol{z}}}$:

$$\Delta(P,Q) := \int \langle |I(\boldsymbol{z}) - I_{\Omega_{\boldsymbol{z}}}|\rangle d\boldsymbol{z}.$$

- Inserting the optimal filter Q^{opt} into $\Delta(P,Q)$ to obtain

Theorem 3. *Let's define, from the incident signal $p(t,s)$, the power spectrum $W(\omega) := P(\omega)\overline{P(\omega)}$. Since W is independent of $\boldsymbol{z}$, it is a solution of the fixed-point functional $K[W] = W$, where K is defined as*

$$K[W] := \begin{cases} 0, & W = 0, \\ \left(\frac{1}{\lambda}\frac{A^2 S_T \sigma_{\eta T}}{(|A|^2 J + \sigma_{\eta T}/W)^2} d\boldsymbol{z}\right)^{-1/2}, & W > 0. \end{cases}$$

Optimal Waveform Spectrum Design

Dem. 2. *We minimize the asymptotic MSE using Lagrange multipliers' method. Let's consider the functional*

$$\Delta_\lambda(W) = \Delta(W, O^{opt}) + \lambda \left(\int W(\omega, s) d\omega ds - M \right).$$

Taking the variational derivative w.r.t. W, we get

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Delta_\lambda(W + \epsilon W_\epsilon) = \int W_\epsilon \frac{-|A|^2 S_\eta}{(|A|^2 JW + S_\eta/S_T)^2} \mathrm{d}\boldsymbol{z} d\omega ds + \lambda \int W_\epsilon \mathrm{d}\omega ds.$$

The r.h.s. must be zero for all W_ϵ, we therefore have

$$\int \frac{|A|^2 S_\eta}{(|A|^2 JW + S_\eta/S_T)^2} \mathrm{d}\boldsymbol{z} = \lambda,$$

for a.e. (s, ω). If the power spectra S_T and S_η are continuous, it holds for every (s, ω).

Full Waveform via Spectral Factorization I

- We need to determine the corresponding **phase information**.
- It's well known that a **minimum-phase waveform** (causal) is uniquely determined from its Fourier transform magnitud via the **Hilbert Transform Method** (Kolmogorov, 1939).
- The strategy is to extend the waveform spectrum to the complex ζ-plane, so that $\omega = \text{Re}\{\zeta\}$, $P(-\zeta, s) = \overline{P(\zeta, s)}$, and $W(\zeta, s) = P(-\zeta, s)P(\zeta, s)$, where all singularities of $P(\zeta, s)$ lie in the upper-half ζ-plane.
- Singularities of $P(\zeta, s)$ being only in the upper half-plane implies that $p(t, s)$ is **causal**.
- We (numerically) construct p as follows. Assuming that W is nonzero, we write (omitting dependence on s)

$$W(\zeta) = e^{\ln W(\zeta)}, \text{ and } P(\zeta) = e^{\ln P(\zeta)}.$$

Full Waveform via Spectral Factorization II

- For $\zeta = \omega \in \mathbb{R}$, $W(\omega) > 0$ is a real-valued, even function of ω, and therefore so is $\ln W(\zeta)$, which can be written as $\ln W(\zeta) = C(-\zeta) + C(\zeta)$ for some C whose singularities lie only in the upper-half plane.

- Therefore, $W(\zeta) = e^{C(-\zeta)+C(\zeta)} = e^{C(-\zeta)}e^{C(\zeta)}$.

- By construction, the inverse FT $c(t)$ of $C(\zeta)$ is a causal function. Consequently, we take $P(\omega) = e^{C(\omega)}$.

- To avoid zero evaluations of logarithm, we regularize our spectrum slightly considering a smoothed spectrum $W_\rho := u_\rho *_\omega W(\omega)$, where u_ρ is the heat kernel $u_\rho := \exp\left[-\omega^2/(2\rho)\right]/\sqrt{2\pi\rho}$.

- The resulting W_ρ will be nonzero for all ω, and we can apply the above Hilbert transform method to it and recover a waveform $p_\rho(t)$, and then

$$p(t, s) = \lim_{\rho \to 0} p_\rho(t, s).$$

Time-evolution optimal waveform $p(t)$

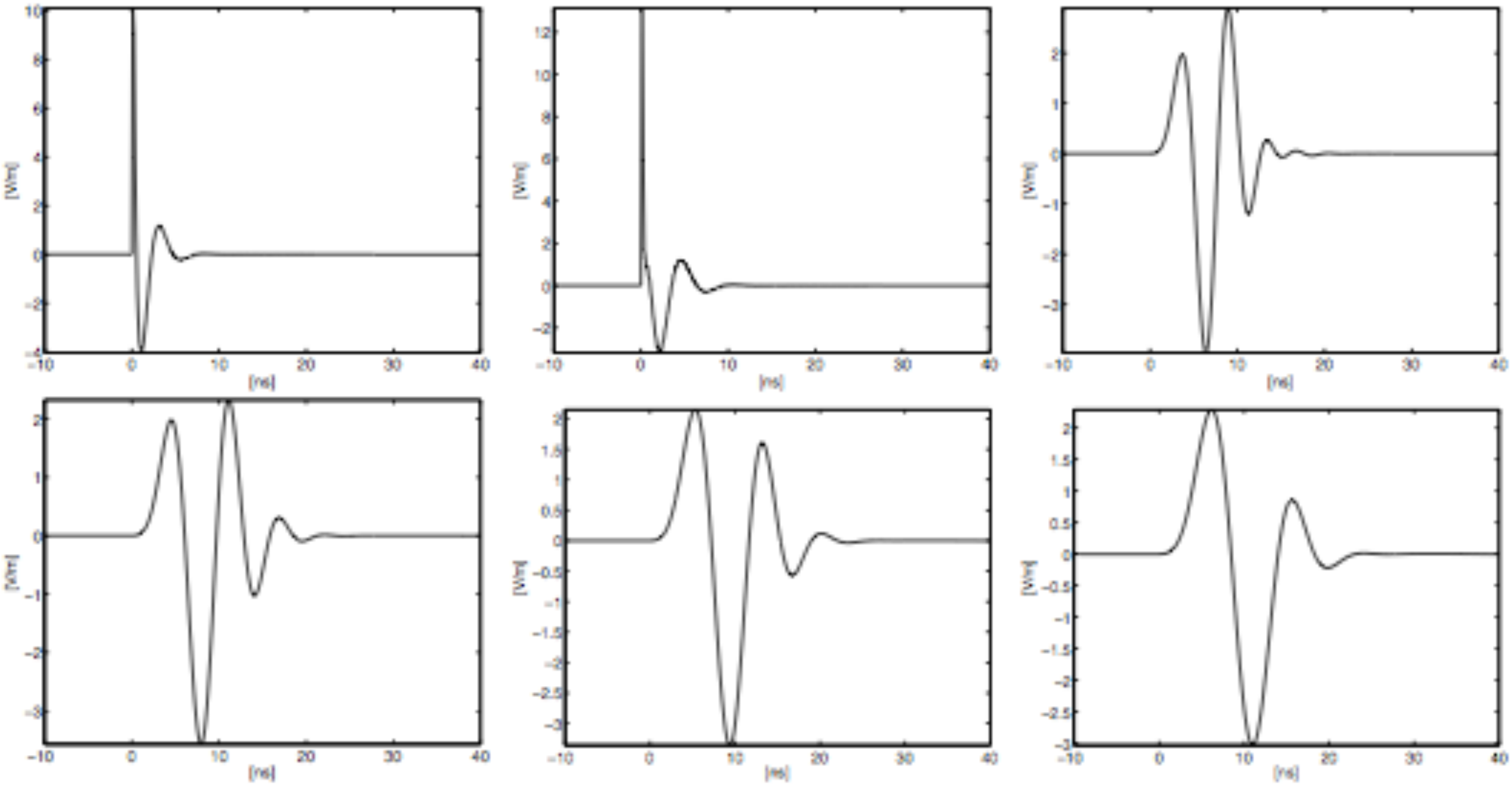


Figure 3: Increasing dispersion from left to right and from the above to the bottom. The optimal waveform adjusts its shape (time-domain) to its environment and, consequently, its performance.

Optimal waveform spectral density

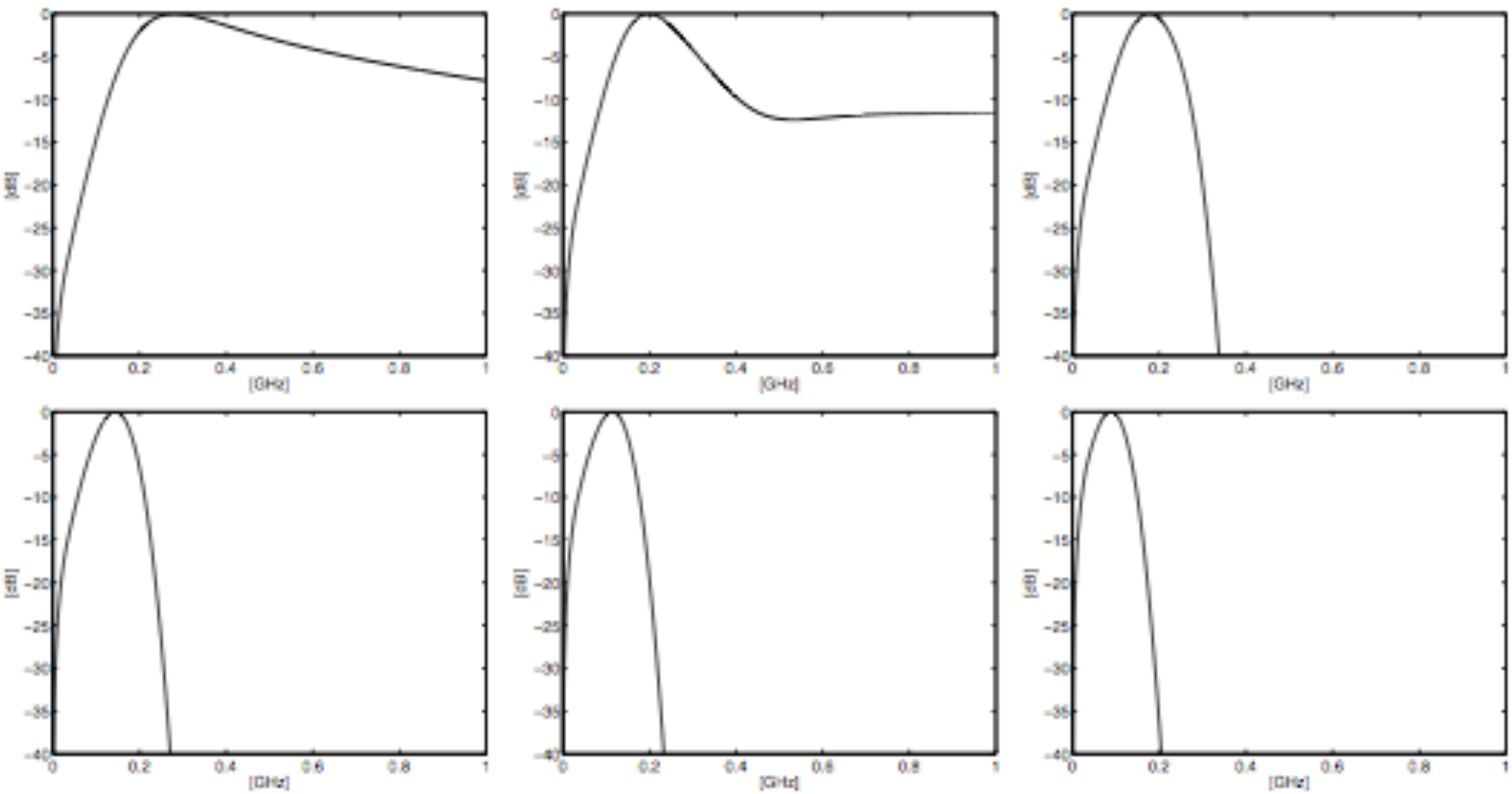


Figure 4: Increasing dispersion from left to right and from the above to the bottom. The optimal waveform adjusts its shape (frequency-domain) to its environment and, consequently, its performance.

Imaging transversal cuts

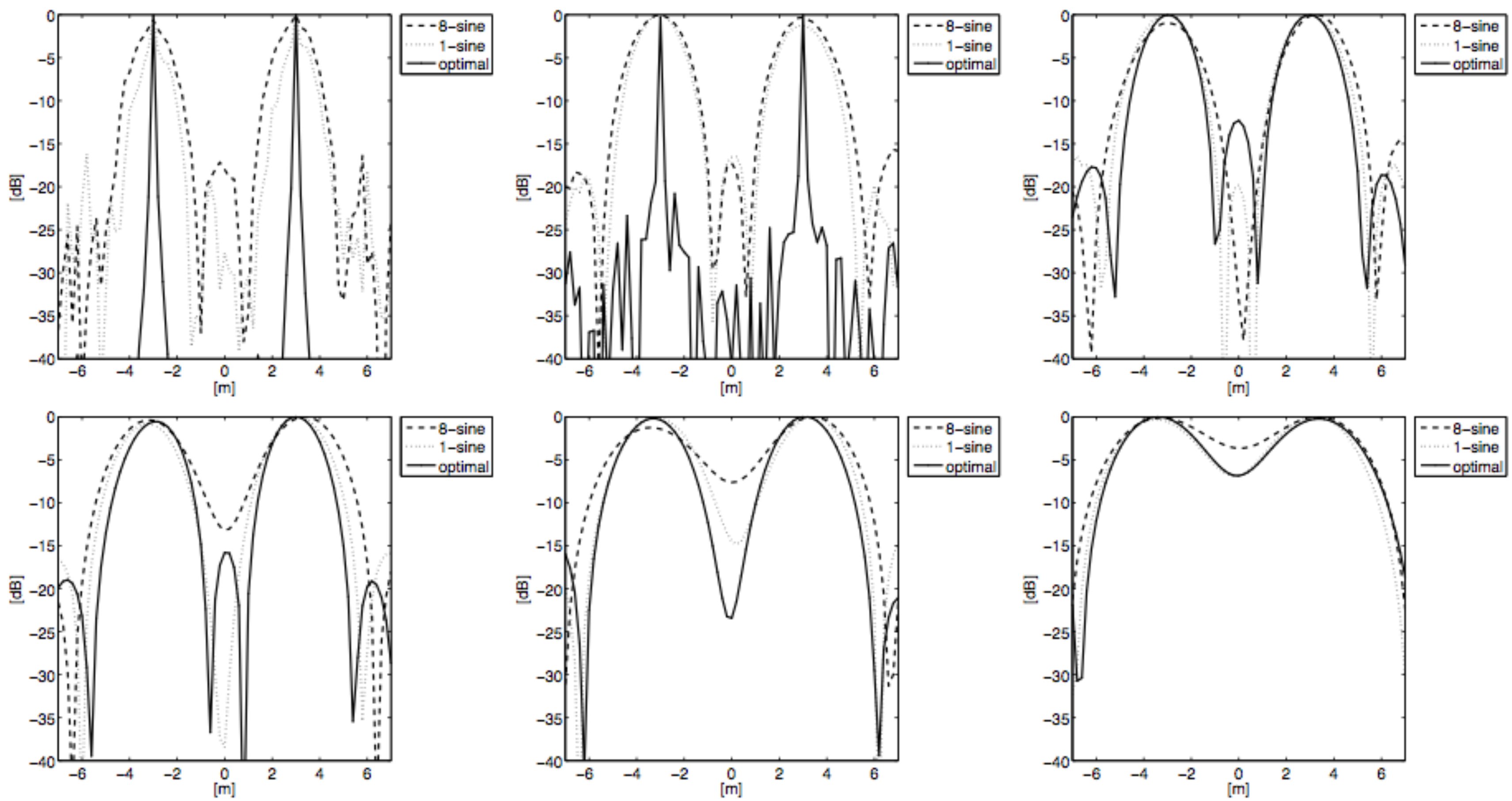


Figure 5: First row the noise is increasing, second row the dispersion is increasing.

Optimal waveform and Brillouin precursor comparisons

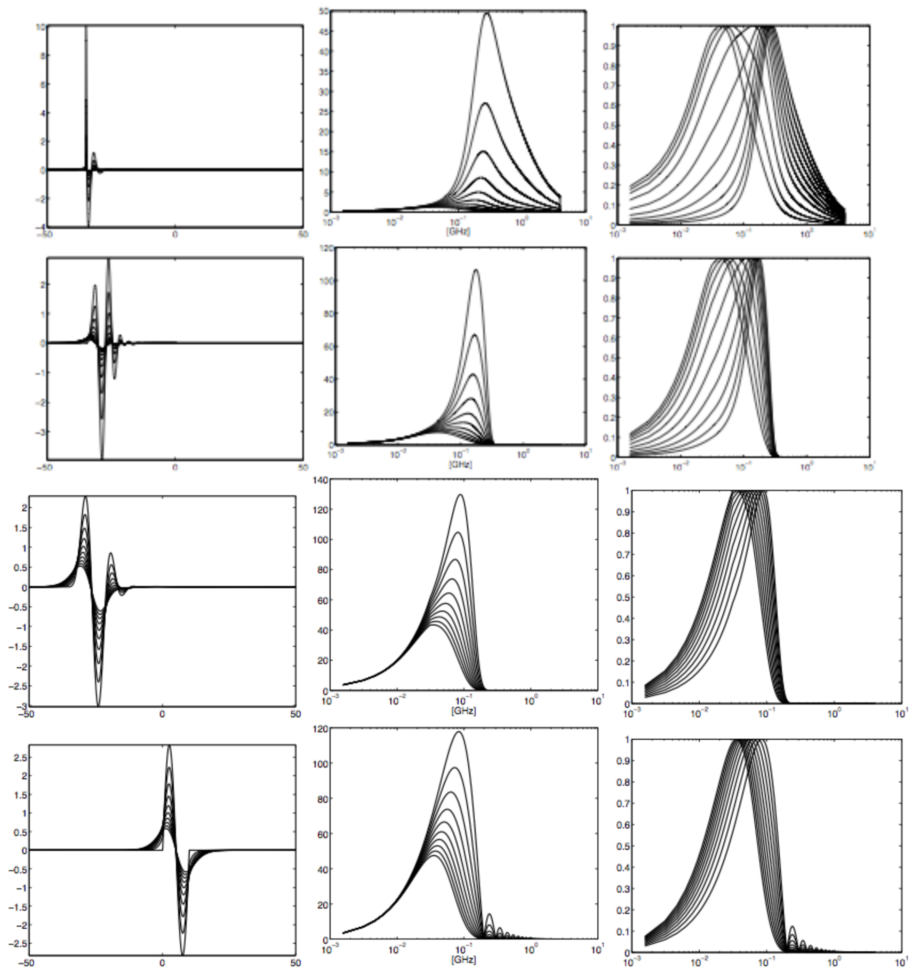


Figure 6: First three rows show optimal waveform evolution, las row shows the Brillouin precursor evolution.

Final Remarks

Challenge: develop a theory for wave propagation in dispersive materials

- Wave propagation in complex and random media:
  - Propagation of cell-phones signals in urban enviroments.
  - Propagation of radar waves through foliage, sea ice, or soil.
  - Propagation of light, microwaves and ultrasound in human tissue.

  For many of these problems, it is the connection between the statistical properties of the enviroment and the wave propagation that one wants to understand.

- **Elastography** (Medical Physics). Optimal waveform: pressure waves or torsional waves? Ivan Rosado, U. Wisconsin-Madison. PhD Thesis Fatima Rodriguez (UAM-I).

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Thank you (all) and

Happy Birthday Steve!

(Nice to meet you)